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MEANWHILE, WITHIN THE FREGE BOUNDARY*

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Abstract

With this squib I want to contribute to understanding and improving upon Keenan's intriguing equivalence result about reducible type $\langle 2\rangle$ quantifiers (Keenan, 1992). I give an alternative proof of his result which generalizes to type $\langle n\rangle$ quantifiers, and I show how the reduction of a reducible type $\langle n\rangle$ quantifier to (the composition of) n type $\langle 1\rangle$ quantifiers can be effectuated.

1 Introduction

Edward Keenan (Keenan 1992) has shown that type $\langle 2\rangle$ quantifiers (properties of binary relations) which are reducible to two type $\langle 1\rangle$ quantifiers (properties of unary relations) are identical if they behave the same on relations which are products. This is remarkable because it allows us to draw universal conclusions about two predicates (over a domain of relations) from their behavior over a highly restricted domain of relations (products, basically). Normally, knowing that two predicates behave uniformly over a small domain (that the nice students are the good students, for instance), does not generalize to larger domains (that nice humans are good humans, a non-sequitur).

Keenan's result is useful because it allows us to actually prove quite a few type $\langle 2\rangle$ quantifiers to be *not* reducible to two type $\langle 1\rangle$ quantifiers. However, the result is not entirely satisfying since it leaves a few questions unanswered. Firstly, Keenan himself already realized that we can not use this result to show, for any irreducible type $\langle 2\rangle$ quantifier, that it is irreducible. Secondly, it does not give us a method for deciding, given the behaviour of a type $\langle 2\rangle$ quantifier on relations which are products, what its possible reduction to two type $\langle 1\rangle$ quantifiers could be. Thirdly, it has so far been unclear if, or how, Keenan's result generalizes to type $\langle n\rangle$ quantifiers, properties of n-ary relations.

In this squib I answer these questions. I will generalize Keenan's result to type $\langle n\rangle$ quantifiers, I will show that if we are given the behaviour of any type $\langle n\rangle$ quantifier on products, we can determine whether it is reducible or not, and, if it is, what are the n type $\langle 1\rangle$ quantifiers to which the type $\langle n\rangle$ one can be reduced. Section 2 states the setting and terminology. Section 3 presents my generalizations of Keenan's reducibility results and section 4 winds up the results.

2 Keenan on type $\langle 2\rangle$ Quantifiers

Let E be our universe of at least two individuals. A type $\langle 1\rangle$ quantifier f is a property of sets of individuals: $f \in \mathcal{P}(\mathcal{P}(E))$, a type $\langle 2\rangle$ quantifier F^2 is a property of binary relations between

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individuals: $F^2 \in \mathcal{P}(\mathcal{P}(E^2))$ and, more generally, a type $\langle n \rangle$ quantifier F^n is a property of n-ary relations over individuals: $F^n \in \mathcal{P}(\mathcal{P}(E^n))$. For a type $\langle n \rangle$ quantifier F^n and $R^n \in \mathcal{P}(E^n)$, I will write $F^n(R^n) = 1$ if $R^n \in F^n$ and $F^n(R^n) = 0$ otherwise.

By means of a rule of division (Geach) a type $\langle m \rangle$ quantifier can also be applied to a relation R of arbitrary arity $n+m$, yielding an n-ary relation $F^m(R)$ as the result:

$F^m(R) =$

$\{\langle d_1, \ldots, d_n \rangle \mid F^m(\{\langle d_{n+1}, \ldots, d_{n+m} \rangle \mid \langle d_1, \ldots, d_{n+m} \rangle \in R\}) = 1\}$

(Notice that if $n = 0$, indeed $F^m(R)$ is either $\{\langle\rangle\}$, the truth value 1, or $\emptyset$, the truth value 0.) Using the rule of division type $\langle 1 \rangle$ quantifiers f and g can be composed to produce a type $\langle 2 \rangle$ quantifier. Thus, $\forall R \in \mathcal{P}(E^2)$:

$(f \circ g)(R) = f(g(R)) = f(\{d \mid g(\{d' \mid \langle d, d' \rangle \in R\}) = 1\})$

For readers familiar with montague grammar, $f \circ g$ is indeed the property of relations R satisfying: $T(\lambda x\, T'(\lambda y\, S(x)(y)))$, with T interpreted as f, T' as g and S as R. For example, consider the composition given by "every cat — a mouse":

$([\![\textit{every cat}]\!] \circ [\![\textit{a mouse}]\!])(R) = 1$ iff

$[\![\textit{cat}]\!] \subseteq \{d \mid [\![\textit{mouse}]\!] \cap \{d' \mid \langle d, d' \rangle \in R\} \neq \emptyset\}$

This type $\langle 2 \rangle$ quantifier holds of any relation R (such as chase, for instance) iff every cat Rs (chases) a mouse.

A both philosophically and linguistically interesting question is concerned with the possibility of characterizing type $\langle 2 \rangle$ quantifiers by means of the composition of two type $\langle 1 \rangle$ quantifiers. (Keenan 1992) presents a number of natural language examples which can not, and he actually proves they are not. The key concept is that of reducibility:

Definition 1 (Reducibility) *A type $\langle 2 \rangle$ quantifier F^2 is reducible iff there are type $\langle 1 \rangle$ quantifiers f and g: $F^2 = f \circ g$.*

If a type $\langle 2 \rangle$ quantifier is not reducible, Keenan has it that it lives beyond the Frege boundary. Keenan's observations are backed up by two theorems, the first one of which we focus upon here:

Theorem 1 (Reducibility Equivalence) *If F^2 and G^2 are reducible type $\langle 2 \rangle$ quantifiers, then $F^2 = G^2$ iff $\forall P, Q \in \mathcal{P}(E)$: $F^2(P \times Q) = G^2(P \times Q)$.*

Reducible quantifiers have the special property that if they behave the same on relations which are products, they behave the same on all relations. (A product $(P \times Q)$ is of course the relation $\{\langle d, d' \rangle \mid d \in P \ \&\ d' \in Q\}$ holding between all members of P and Q, respectively.) This is an intriguing and remarkable result which can be used to show certain type $\langle 2 \rangle$ quantifiers to be not reducible.

Consider an arbitrary type $\langle 2 \rangle$ quantifier and the question whether it is reducible or not. Of course, if we take $[\![\textit{every cat}]\!] \circ [\![\textit{a mouse}]\!]$ we know it is reducible because the type $\langle 2 \rangle$ quantifier is defined in terms of two type $\langle 1 \rangle$ quantifiers. But then consider a property like that of transitivity or reflexivity. Transitivity and reflexivity are (contingent) properties of relations so they are type $\langle 2 \rangle$ quantifiers as defined above. Can we define these properties using two type

$\langle 1\rangle$ quantifiers? Now one may try to do this, and one may fail to succeed in reducing these quantifiers but this does not need to show that they are not reducible. Maybe one has not tried hard enough! Keenan here offers an ingenious method to establish that these quantifiers are indeed not reducible. Consider what transitivity and reflexivity say about product relations. It turns out that:

$\text{TRANS}(P \times Q) = 1\ (\forall P, Q \in \mathcal{P}(E))$

$\text{REFL}(P \times Q) = 1 \text{ iff } P = Q = E$

This means that transitivity and reflexivity display precisely the same truth value pattern on product relations as the type $\langle 2\rangle$ quantifiers $(\top \circ \top)$ and $(\text{ALL} \circ \text{ALL})$, respectively. (Here, $\top$ is the type $\langle 1\rangle$ quantifier true of all sets of individuals.) Notice that the latter two type $\langle 2\rangle$ quantifiers are reducible, because they are each defined in terms of two type $\langle 1\rangle$ quantifiers. With Keenan's theorem (1) we now know that if transitivity and reflexivity are reducible then $\text{TRANS} = (\top \circ \top)$ and $\text{REFL} = (\text{ALL} \circ \text{ALL})$. But since the latter two equations are definitely false, the assumptions that transitivity and reflexivity are reducible must be false as well. A proof of the non-reducibility of a type $\langle 2\rangle$ quantifier F^2 thus consists in defining a type $\langle 2\rangle$ quantifier $f \circ g$ which behaves the same as F^2 on product relations. If $f \circ g$ is not in general equal to F^2 we know F^2 to be not reducible.

Before we proceed, let us look at three natural language examples.

(1) Lois and Clark posed the same two stupid questions.

(2) Every student criticised himself.

(3) A sum total of five theories handled a sum total of five sentences.

If we only look at models where "posed" may denote relations which are products $P \times Q$, then (1) is true iff (i) Lois and Clark are in P and (ii) there are exactly two questions in Q. But these are precisely the same products for which "Lois and Clark posed exactly two stupid questions" is true. Since $([\![Lo{+}Cl]\!] \circ [\![ex2stqu]\!])$ is reducible and not equal to $\{R \mid [\![Lo{+}Cl\ V\ sa2stqu]\!]_{V/R} = 1\}$, the latter is not reducible. Similarly, only looking at product interpretations of "criticised", example (2) is true iff "Every student criticised every student" is true, but certainly the two sentences are not generally equivalent. The same finally goes for example (3) (on the cumulative reading) and "Exactly five theories handled exactly five sentences." These observations thus show that the examples (1)–(3) cannot be analyzed (compositionally) as involving a relation and two type $\langle 1\rangle$ quantifiers. See (Keenan 1992) for more discussion.

Keenan's Reducibility Equivalence is a truly interesting result, but it leaves us with a couple of questions. Firstly, it is not quite clear exactly why reducible type $\langle 2\rangle$ quantifiers behave as Keenan's theorem says they do. What makes it that their behaviour on the full domain $\mathcal{P}(E^2)$ is, in a sense, determined by their behaviour on $\mathcal{P}(E) \times \mathcal{P}(E)$? I must submit that, although I could follow Keenan's own proof of theorem (1), it did not give me the feeling I could see what is at stake. (Notice that it is certainly not the case that $(f \circ g)(R) = (f(d(R)) \wedge g(r(R)))$, where $d(R)$ indicates the domain of R and $r(R)$ its range.) Secondly, Keenan's theorem is only partly helpful in proving non-reducibility. For to prove type $\langle 2\rangle$ F^2 not reducible we still have to find a (different) quantifier $(f \circ g)$ which behaves the same as F^2 on products. But if we do not find such a composition of two type $\langle 1\rangle$ quantifiers it at best shows that F^2 is not reducible or, again, we have not tried hard enough! Besides, as we will see below, there are type $\langle 2\rangle$ quantifiers, viz., the property of being a symmetric relation, the behaviour of which on products can not be characterized by any reducible quantifier. Thirdly, it has so far been an open question whether Keenan's reducibility equivalence generalizes to type $\langle n\rangle$ quantifiers. The following section is devoted to answer these questions.

3 Generalizing Keenan's Result

Let us first generalize our notion of reducibility:

Definition 2 (Type $\langle n \rangle$ Reducibility) *A type $\langle n \rangle$ quantifier F^n is (n)-reducible iff there are n type $\langle 1 \rangle$ quantifiers $f_1, \dots, f_n$: $F^n = f_1 \circ \dots \circ f_n$.*

One of the key concepts which Keenan also uses is that of a quantifier which is 'positive'. A quantifier F^n (of arbitrary type $\langle n \rangle$) is positive iff $F^n(\emptyset) = 0$. Our observations in this squib will be stated for the most part with respect to positive quantifiers and with respect to type $\langle n \rangle$ quantifiers which are reducible to n positive type $\langle 1 \rangle$ quantifiers, without loss of generalization. For:

Observation 1 *If F^n is an (n)-reducible type $\langle n \rangle$ quantifier then there are n positive type $\langle 1 \rangle$ quantifiers $f_1, \dots, f_n$ such that $F^n = f_1 \circ \dots \circ f_n$ or $F^n = \neg f_1 \circ \dots \circ f_n$.*

Proof. Suppose F^n is (n)-reducible so that $F^n = f_1 \circ \dots \circ f_n$. Starting from $i = n$ up to $i = 1$, if f_i is not positive, use $\neg f_i$ instead, which is positive, and, if $i > 1$, use $f_{i-1}\neg$ in stead of f_{i-1}. Obviously, $f_{i-1}\neg \circ \neg f_i = f_{i-1} \circ f_i$. This, thus, is a recipe for characterizing an (n)-reducible type $\langle n \rangle$ quantifier F^n or $\neg F^n$ by means of n positive type $\langle 1 \rangle$ quantifiers. We will also use a generalization of the following observation from Keenan:

Observation 2 *For a positive type $\langle 1 \rangle$ quantifier f and any $P, Q \in \mathcal{P}(E)$: $f(P \times Q) = P$ if $f(Q) = 1$ and $f(P \times Q) = \emptyset$ otherwise.*

Proof. If $d \notin P$, $d \notin f(P \times Q)$, since f is positive; if $d \in P$, $d \in f(P \times Q)$ iff $f(Q) = 1$. The generalization we use is this:

Observation 3 *If $F^n = f_1 \circ \dots \circ f_n$ and the f_i are positive, then $F^n(Q_1 \times \dots \times Q_n) = 1$ iff $f_1(Q_1) = \dots = f_n(Q_n) = 1$.*

Proof. Assuming that $f_1(Q_1) = \dots = f_n(Q_n) = 1$, $n-1$ applications of observation (2) give us that $F^n(Q_1 \times \dots \times Q_n) = (f_1 \circ \dots \circ f_n)(Q_1 \times \dots \times Q_n) = (f_1 \circ \dots \circ f_{n-1})(Q_1 \times \dots \times Q_{n-1}) = \dots = f_1(Q_1) = 1$. Furthermore, if, for any i $(1 < i \leq n)$ $f_i(Q_i) = 0$, $F^n(Q_1 \times \dots \times Q_n) = (f_1 \circ \dots \circ f_n)(Q_1 \times \dots \times Q_n) = f_1(\emptyset) = 0$ (because the f_i are positive), and otherwise, if only $f_1(Q_1) = 0$, $(f_1 \circ \dots \circ f_n)(Q_1 \times \dots \times Q_n) = f_1(Q_1) = 0$ as well.

Now suppose F^n and G^n are (n)-reducible type $\langle n \rangle$ quantifiers. We can for the sake of convenience assume that $F^n = f_1 \circ \dots \circ f_n$ and $G^n = g_1 \circ \dots \circ g_n$, with all of the f_i and g_j positive. (Otherwise, use $\neg F^n$ and/or $\neg G^n$, cf. observation 1). Keenan's theorem is now easily generalized:

Theorem 2 (Type $\langle n \rangle$ Reducibility Equivalence) *If F^n and G^n are type $\langle n \rangle$ quantifiers (n)-reducible to positive type $\langle 1 \rangle$ quantifiers, then $F^n = G^n$ iff $\forall Q_1, \dots, Q_n \in \mathcal{P}(E)$: $F^n(Q_1 \times \dots \times Q_n) = G^n(Q_1 \times \dots \times Q_n)$.*

Proof. Let F^n be reducible so that $F^n = f_1 \circ \ldots \circ f_n$ with all of the f_i positive. This means $F^n(Q_1 \times \ldots \times Q_n) = 1$ iff $f_i(Q_i) = 1$ for all i: $1 \leq i \leq n$. The same goes for $G^n = g_1 \circ \ldots \circ g_n$, with the g_j positive. If F^n and G^n behave the same on products, the f_i must be identical to the g_i so that $F^n = G^n$. (Obviously, if $F^n = G^n$, they behave the same on products.)

Keenan's findings about (2)-reducible type $\langle 2 \rangle$ quantifiers are thus generalized to type $\langle n \rangle$. The behaviour of n-reducible type $\langle n \rangle$ quantifiers on arbitrary n-ary relations is somehow determined by their behaviour on relations which are products of n sets of individuals. An obvious next question is this. Given the behaviour of a quantifier F^n on products, can we determine what, if any, are type $\langle 1 \rangle$ quantifiers $f_1, \ldots, f_n$ such that $F^n = f_1 \circ \ldots \circ f_n$? We can, if F^n shows a certain invariance, defined as follows:

Definition 3 (Invariance) *A type $\langle n \rangle$ quantifier F^n is invariant for sets in products iff $\forall Q_1, \ldots, Q_n, Q_1', \ldots, Q_n'$ (all non-empty) and for any i $(1 \leq i \leq n)$:*

if $F^n(Q_1 \times \ldots \times Q_i \times \ldots \times Q_n) = F^n(Q_1' \times \ldots \times Q_i' \times \ldots \times Q_n') = 1$

then $F^n(Q_1 \times \ldots \times Q_i' \times \ldots \times Q_n) = 1$.

If we are given the behaviour of a type $\langle n \rangle$ quantifier on products we can now determine whether that behaviour can be generated by n type $\langle 1 \rangle$ quantifiers. The point is not that F^n is invariant iff F^n is reducible, but the idea comes close:

Theorem 3 (Reducible Product Equivalents) *A type $\langle n \rangle$ quantifier F^n or $\neg F^n$ is invariant for sets in products iff there is a product equivalent (n)-reducible correlate G^n of F^n.*

Proof, Only if. Suppose F^n is invariant for sets in products. Define, for non-empty Q_i: $g_1(Q_1) = \ldots = g_n(Q_n) = 1$ iff $F^n(Q_1 \times \ldots \times Q_n) = 1$, $g_2(\emptyset) = \ldots = g_n(\emptyset) = 0$ and $g_1(\emptyset) = F^n(\emptyset)$. By F^n's invariance this is well-defined. Take $G^n = g_1 \circ \ldots \circ g_n$. By its definition G^n is equivalent with F^n on product relations and (n)-reducible. Furthermore, if $\neg F^n$ is invariant, construct the correlate $G^n = g_1 \circ \ldots \circ g_n$ of $\neg F^n$ and then $\neg G^n = \neg g_1 \circ \ldots \circ g_n$ is the reducible product equivalent of F^n.

If. Let $G^n = g_1 \circ \ldots \circ g_n$ be a product equivalent (n)-reducible correlate of F^n. Using observation (1) we can assume the g_i to be positive for $1 < i \leq n$. First assume g_1 is positive as well. Then $F^n(Q_1 \times \ldots \times Q_n) = F^n(Q_1' \times \ldots \times Q_n') = 1$ iff (product equivalence) $G^n(Q_1 \times \ldots \times Q_n) = G^n(Q_1' \times \ldots \times Q_n') = 1$ iff (observation 3) $g_i(Q_i) = g_i(Q_i') = 1$, for any $1 \leq i \leq n$. But then $G^n(Q_1 \times \ldots \times Q_i' \times \ldots \times Q_n) = 1$ (observation 3) and $F^n(Q_1 \times \ldots \times Q_i' \times \ldots \times Q_n) = 1$ (product equivalence). Hence, F^n is invariant. Now assume g_1 is not positive. Then $\neg G^n$ is the composition $\neg g_1 \circ \ldots \circ g_n$ of n positive quantifiers which is product equivalent with $\neg F^n$ and we can use the very same method to show that $\neg F^n$ is invariant.

Theorem (3) tells us, when we know the behaviour of a type $\langle n \rangle$ quantifier $(\neg)F^n$ on products, we know whether there is an (n)-reducible quantifier which has that behaviour. Moreover, the proof above gives us a method for defining this reducible quantifier as the composition of n constructively defined type $\langle 1 \rangle$ quantifiers. Thus we can sharpen our findings about reducibility:

Corollary 1 (Decomposition) *If a type $\langle n \rangle$ quantifier F^n is invariant for sets in products, then F^n is (n)-reducible iff $F^n = G^n = g_1 \circ \ldots \circ g_n$, with the g_i defined as in the proof of theorem (3).*

Proof. If F^n is invariant it has a reducible product equivalent correlate G^n (theorem 3) and if F^n is reducible as well it must be identical with G^n (theorem 2). Theorem (3) also helps us further in proving non-reducibility, for:

Corollary 2 (Non-reducibility) *If a type $\langle n\rangle$ quantifier F^n and its negation are not invariant for sets in products, then F^n is not (n)-reducible.*

Proof, by contraposition. Suppose F^n is (n)-reducible. Then F^n has a product equivalent correlate G^n, namely F^n itself, which is (n)-reducible by supposition. So, by theorem (3), we find that F^n or $\neg F^n$ is invariant for sets in products.

The previous observations give us a precise method for establishing reducibility results. Given a type $\langle n\rangle$ quantifier F^n, first check whether F^n and $\neg F^n$ are invariant for sets in products. If they are not invariant, they are not reducible (corrolary 2). If one of them is, then construct the reducible product equivalent correlate G^n of F^n (theorem 3) and check whether $F^n = G^n$. If they are not the same, F^n is not reducible (theorem 2); if they are the same, then, of course, F^n is reducible.

4 Conclusion

With this squib I have hoped to contribute to understanding Keenan's result from (Keenan 1992). Reducible type $\langle 2\rangle$ quantifiers that behave the same on product relations are the same. I have given an alternative proof of this result, which applies to type $\langle n\rangle$ quantifiers in general. Not only is this a new and welcome generalization, it also gives some insight into the intimate relation between (n)-reducible type $\langle n\rangle$ quantifiers and n-ary product relations. If type $\langle n\rangle$ quantifier F^n is (n)-reducible, that is, if $F^n = f_1 \circ \ldots \circ f_n$ (with the f_i positive), then F^n is satisfied by $Q_1 \times \ldots \times Q_n$ iff each composing f_i is satisfied by Q_i.

Corollary (1) and the construction used in the proof of theorem (3) furthermore prove useful if we want to use Keenan's theorem (1) to establish non-reducibility results. Take transitivity and reflexivity again. Transitivity is true of all products and trivially invariant for sets in products. The construction used in the proof of theorem (3) automatically gives us $G^2 = (\top \circ \textsc{some})$ as the one and only (2)-reducible type $\langle 2\rangle$ quantifier which behaves thus on products. (Of course, $(\top \circ \textsc{some}) = (\top \circ g)$ for arbitrary g.) Since $\textsc{trans} \neq (\top \circ \textsc{some})$, we know transitivity is not reducible. Reflexivity holds only on the product $E \times E$, so it is invariant, too. The reducible product equivalent correlate of $\textsc{refl}$ is $G^2 = (g_1 \circ g_2)$, with $g_1(Q) = g_2(Q') = 1$ iff $Q = Q' = E$. This is indeed the reducible quantifier $(\textsc{all} \circ \textsc{all})$, different from $\textsc{refl}$. So reflexivity is not reducible either.

Theorem (3) also helps us settle the matter about type $\langle 2\rangle$ quantifiers such as $\textsc{symm}$. Products $(P \times Q)$ are symmetric iff $P = Q$ or one of them is empty. But certainly $\textsc{symm}$ is not invariant for sets in products in $\textsc{symm}$: $\textsc{symm}(P \times P) = \textsc{symm}(Q \times Q) = 1$ while $\textsc{symm}(P \times Q) = \textsc{symm}(Q \times P) = 0$ if $\emptyset \neq P \neq Q \neq \emptyset$. Thus, $\textsc{symm}$ and $\neg\textsc{symm}$ are not invariant and theorem (3) tells us that there is no (2)-reducible type $\langle 2\rangle$ quantifier with the same behaviour as (non-)symmetry on products. This explains why we can not use Keenan's theorem (1) to show symmetry not to be reducible. And it also explains why we do not at all need theorem (1) for that purpose. Theorem (3), or, rather, corollary (2), already tells us that it is not reducible. The generalization of Keenan's theorem presented in this squib not only improves our understanding of it, but it also extends its range of application.

References

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