MEASUREMENT-THEORETIC FOUNDATIONS OF INTERADJECTIVE-COMPARISON LOGIC*

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1 Introduction

By means of comparative constructions, we can compare individuals according to different properties. Such comparisons may be divided into two types: *direct* and *indirect* comparisons. The former are comparisons of *direct measurements*. As an example of them, we can give *interadjective comparatives* like the following:[1]

1(a) Albert is taller than Charles is wide.

The latter are comparisons of *relative positions on different scales* as in:

1(b) Albert is taller than Charles is intelligent.

Traditionally, indirect comparisons have been treated separately from direct comparisons.[2] Bale (2008) proposes a unified theory of direct and indirect comparisons. van Rooij (2011) criticises Bale's theory in terms of *measurement theory*. According to van Rooij, Bale's theory gives rise to *ratio scales*, which provide truth conditions for comparatives like the following:

1(c) Albert is five times as tall as Charles is intelligent.

According to van Rooij, 1(c) is as *meaningless* as the following *interpersonal comparison* of utility in social choice theory:

Action x is five times as useful for John as action y is for Mary.

On the other hand, the following is meaningful:

1(d) Albert is three times as tall as Charles is wide.

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[1] In (Suzuki, 2011b) we propose a logic called ICL designed especially for gradable adjectives.

[2] For example, (Cresswell, 1976).

There are two main problems with measurement theory[3]:

1. the *representation problem*—justifying the assignment of numbers to objects, and
2. the *uniqueness problem*—specifying the transformation up to which this assignment is unique.

A solution to the former can be furnished by a *representation theorem*, which establishes that the specified conditions on a qualitative relational system are (necessary and) sufficient for the assignment of numbers to objects that represents (preserves) all the relations in the system. A solution to the latter can be furnished by a *uniqueness theorem*, which specifies the transformation up to which this assignment is unique. In this paper, we provide truth conditions for interadjective comparatives like the followings:

1(e) Albert is 10 centimetres taller than Catherin is wide.

1(f) Albert is taller than Bernard by more than Charles is wider than Dennis.

1(g) Albert is taller than Bernard by more than Charles is more intelligent than Dennis.

In terms of constraints on transformations, measurement theorists distinguish among scales: absolute, ratio, interval, ordinal, nominal, log-interval, and difference scales.[4] These scales are based on the assignment of a measure on a set of individuals (domain). In order to make sense of interadjective comparisons, van Rooij (2011) thinks of the domain not just as a set of individuals but rather as a set of individual-adjective pairs. He calls *co-ordinal scales* ordinal scales based on the assignment of a measure on a set of individual-adjective pairs, *co-interval scales* interval scales based on it, and co-ratio scales ratio scales based on it. He argues that co-ordinal scales can make comparatives like 1(a) and 1(b) meaningful, and that co-interval scales can account for examples like 1(e), 1(f), and 1(g). Moreover, he suggests that co-ratio scales can make comparatives like 1(d) meaningful. Klein (1980) analyses comparatives by existentially quantifying over the meaning of *modifiers of adjectives* like *very* and *fairly*. This analysis can provide 1(a) with the following truth condition:

$$\exists f \in \{very, fairly, quite, \ldots\}(f(tall)(Albert) \wedge \neg f(wide)(Charles)).$$

Klein's analysis gives rise only to a co-ordinal scale. van Rooij (2011) tries to extend this analysis to co-interval scales and co-ratio scales. If f is *total* and so applicable to *any* adjective, this analysis then makes all interadjective comparatives meaningful. So van Rooij acknowledges that it is *partial*. But his approach has such a defect that he does not show how to construct this partial function f. Our strategy in this paper is as follows. We prove, by means of the *representation theorem for interadjective-comparison ordering* and the *uniqueness theorem for it*, that *log-interval scales* can make comparatives like 1(b) and 1(g) meaningful, and, by means of the *representation*

[3]Roberts (1979) gives a comprehensive survey of measurement theory. The mathematical foundation of measurement had not been studied before Hölder (1901) developed his axiomatisation for the measurement of mass. (Krantz et al., 1971), (Suppes et al., 1989) and (Luce et al., 1990) are seen as milestones in the history of measurement theory.

[4]We describe these scales later.

theorem for magnitude estimation and the *uniqueness theorem for it*, that *ratio scales* can account for examples like 1(a), 1(d), 1(e), and 1(f). Moreover, we propose conditions under which a model of interadjective comparison and magnitude estimation which gives rise to *ratio scales* can give *numerical assignments*. The semantic structure of 1(c) cannot satisfy some of these conditions. Therefore, we cannot provide 1(c) with its truth condition.

The aim of this paper is to propose a new version of logic for interadjective comparisons–Interadjective-Comparison Logic (ICL). In terms of measurement theory, the *model* of the language of ICL can provide reasonable conditions for log-interval scales and ratio scales, and give the *truth conditions* of interadjective comparatives like 1(a), 1(b), 1(c), 1(d), 1(e), 1(f) and 1(g). Moreover, this model can render interadjective comparatives like 1(c) *meaningless*.

The structure of this paper is as follows. We define the language $\mathscr{L}_{\mathsf{ICL}}$ of ICL, give descriptions of meaningfulness, scale types, magnitude estimation, and the relation between cross-modality matching and interadjective comparison, define a model $\mathfrak{M}$ of $\mathscr{L}_{\mathsf{ICL}}$, formulate the representation theorem for interadjective-comparison ordering and the uniqueness theorem for it, formulate the representation theorem for magnitude estimation and the uniqueness theorem for it, provide ICL with a satisfaction definition, a truth definition, and a validity definition, touch upon the non first-order axiomatisability of models of $\mathscr{L}_{\mathsf{ICL}}$, give the truth conditions of the examples 1(a), 1(b), 1(d), 1(e), and 1(f), and give an answer to the question as to why 1(c) is meaningless.

2 Interadjective-Comparison Logic ICL

2.1 Language

We define the language $\mathscr{L}_{\mathsf{ICL}}$ of ICL as follows:

Definition 1 (Language) *Let $\mathscr{V}$ denote a set of individual variables, $\mathscr{C}$ a set of individual constants, $\mathscr{P}$ a set of n one-place predicate symbols. The language $\mathscr{L}_{\mathsf{ICL}}$ of ICL is given by the following rule:*

$$t ::= x \mid a,$$
$$\varphi ::= P_i(t) \mid t_i = t_j \mid \mathbf{DER}_{P_i,P_j}(t_i,t_j) \mid \mathbf{IER}_{P_i,P_j}(t_i,t_j) \mid$$
$$\mathbf{TER}_{P_i,P_j}{}^k(t_i,t_j) \mid \mathbf{UER}_{P_i,P_j}{}^k(t_i,t_j) \mid \mathbf{FDER}_{P_i,P_j}(t_i,t_j,t_l,t_m) \mid$$
$$\mathbf{FIER}_{P_i,P_j}(t_i,t_j,t_l,t_m) \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\varphi,$$
where $x \in \mathscr{V}$, $a \in \mathscr{C}$, and $P_i, P_j \in \mathscr{P}$.

- $\mathbf{DER}_{P_i,P_j}(t_i,t_j)$ *means that t_i is <u>directly</u> P_i-er than t_j is P_j.*
- $\mathbf{IER}_{P_i,P_j}(t_i,t_j)$ *means that t_i is <u>indirectly</u> P_i-er than t_j is P_j.*
- $\mathbf{TER}_{P_i,P_j}{}^k(t_i,t_j)$ *means that t_i is <u>directly</u> k-times as P_i as t_j is P_j.*
- $\mathbf{UER}_{P_i,P_j}{}^k(t_i,t_j)$ *means that t_i is <u>directly</u> P_i-er than t_j is P_j by n-units of measurement (e.g. (centi)metre, (kilo)gram, ...).*
- $\mathbf{FDER}_{P_i,P_j}(t_i,t_j,t_l,t_m)$ *means that t_i is <u>directly</u> P_i-er than t_j by more than t_l is P_j-er than t_m.*
- $\mathbf{FIER}_{P_i,P_j}(t_i,t_j,t_l,t_m)$ *means that t_i is <u>indirectly</u> P_i-er than t_j by more than t_l is P_j-er than t_m.*

The set of all well-formed formulae of $\mathscr{L}_{\mathsf{ICL}}$ is denoted by $\Phi_{\mathscr{L}_{\mathsf{ICL}}}$.

2.2 Semantics

2.2.1 Meaningfulness and scale types

Roberts (1979:p.52, pp.57–59) argues the *meaningfulness* of sentences involving *scales*. He begins with the the following sentences and considers which seem to be meaningful.

> 2(a) The number of cans of corn in the local super market at closing time yesterday was at least 10.
> 2(b) One can of corn weighs at least 10.
> 2(c) One can of corn weighs twice as much as a second.
> 2(d) The temperature of one can of corn at closing time yesterday was twice as much as that of a second time.

2(a) seems to be meaningful, but 2(b) does not, for the number of cans is specified without reference to a particular scale of measurement, whereas the weight of a can is not. Similarly, 2(c) seems to be meaningful, but 2(d) does not, for the ratio of weights is the same regardless of measurement used, whereas that of temperature is not necessarily the same. Meaningfulness can be studied by analysing the following *admissible transformations* of scale defined by the concept of a *homomorphism*:

Definition 2 (Homomorphism) *Suppose a relational system $\mathfrak{U} := (A, R_1, R_2, \ldots, R_{\mathrm{p}}, \circ_1, \circ_2, \ldots, \circ_{\mathrm{q}})$ and another $\mathfrak{B} := (B, R_1{}', R_2{}', \ldots, R_{\mathrm{p}}{}', \circ_1{}', \circ_2{}', \ldots, \circ_{\mathrm{q}}{}')$, where A and B are sets, $R_1, R_2, \ldots, R_{\mathrm{p}}$ are relations on A, $R_1{}', R_2{}', \ldots, R_{\mathrm{p}}{}'$ are relations on B, $\circ_1, \circ_2, \ldots, \circ_{\mathrm{q}}$ are operations on A, and $\circ_1{}', \circ_2{}', \ldots, \circ_{\mathrm{q}}{}'$ are operations on B. f is called a homomorphism from $\mathfrak{U}$ into $\mathfrak{B}$ if, for any $a_1, a_2, \ldots, a_{\mathrm{r_i}} \in A$,*

$$R_{\mathrm{i}}(a_1, a_2, \ldots, a_{\mathrm{r_i}}) \text{ iff } R_{\mathrm{i}}{}'(f(a_1), f(a_2), \ldots, f(a_{\mathrm{r_i}})),\ i = 1, 2, \ldots, p,$$

and for any $a, b \in A$,

$$f(a \circ_{\mathrm{i}} b) = f(a) \circ_{\mathrm{i}}{}' f(b),\ i = 1, 2, \ldots, q.$$

Definition 3 (Admissible Transformation of Scale) *Suppose that a scale f is one homomorphism from a relational system $\mathfrak{U}$ into another $\mathfrak{B}$, and suppose that A is the set underlying $\mathfrak{U}$ and B is the set underlying $\mathfrak{B}$. Suppose that Φ is a function that maps the range of f, that is, the set $f(A) := \{f(a) : a \in A\}$ into B. Then the composition $\Phi \circ f$ is a function from A into B. If $\Phi \circ f$ is a homomorphism from $\mathfrak{U}$ into $\mathfrak{B}$, we call Φ an admissible transformation of f*

The following provides an example:

Example 1 *Suppose $\mathfrak{U} := (\mathbb{N}, >)$, $\mathfrak{B} := (\mathbb{R}, >)$, and $f : \mathbb{N} \to \mathbb{R}$ is given by $f(x) := 2x$. Then f is a homomorphism from $\mathfrak{U}$ into $\mathfrak{B}$. If $\Phi(x) := x + 5$, then $\Phi \circ f$ is a homomorphism from $\mathfrak{U}$ into $\mathfrak{B}$, for we have $(\Phi \circ f)(x) = 2x + 5$, and*

$$x > y \text{ iff } 2x + 5 > 2y + 5.$$

Thus, $\Phi : f(A) \to B$ is an admissible transformation of f. However, if $\Phi(x) := -x$ for any $x \in f(A)$, then Φ is not an admissible transformation, for $\Phi \circ f$ is not a homomorphism from $\mathfrak{U}$ into $\mathfrak{B}$.

We define meaningfulness in terms of admissible transformations as follows:

Definition 4 (Meaningfulness) *A sentence involving scales is meaningful iff the truth or falsity is unchanged under admissible transformations of all the scales in question.*

Roberts (1979:pp.64–67) defines *scale types* in terms of the class of admissible transformations as follows:

1. The simplest example of a scale is where only admissible transformation is $\Phi(x) = x$. Such a scale is called an *absolute scale*. Counting is an example of an absolute scale.
2. When the admissible transformations are all the functions $\Phi : f(A) \to B$ of the form $\Phi(x) = \alpha x,\ \alpha > 0$, Φ is called a *similarity transformation*, and a scale with the similarity transformations as its class of admissible transformations is called a *ratio scale*. Mass and temperature on the Kelvin scale are examples of ratio scales. According Stevens (Stevens, 1957), various *sensations* such as loudness and brightness can also be measured in ratio scales.
3. When the admissible transformations are all the functions $\Phi : f(A) \to B$ of the form $\Phi(x) = \alpha x + \beta,\ \alpha > 0$, Φ is called a *positive linear transformation*, and a corresponding scale is called an *interval scale*. Temperature on the Fahrenheit scale and temperature on the Celsius scale are examples of interval scales.
4. When a scale is unique up to *order*, the admissible transformations are monotone increasing functions $\Phi(x)$ satisfying the condition that $x \geq y$ iff $\Phi(x) \geq \Phi(y)$. Such scales are called *ordinal scales*. The Mohs scale of hardness is an example of an ordinal scale.
5. In some scales, all one-to-one functions Φ define admissible transformations. Such scales are called *nominal scales*. Examples of nominal scales are numbers on the uniforms of baseball players.
6. A scale is called a *log-interval scale* if the admissible transformations are functions of the form $\Phi(x) = \alpha x^{\beta},\ \alpha, \beta > 0$. Log-interval scales are important in *psychophysics*, where they are considered as scale types for the *psychophysical laws* relating a physical quantity (for example, intensity of a sound) to psychological quantity (for example, loudness of a sound).
7. When the admissible transformations are functions of the form $\Phi(x) = x + \beta$, a corresponding scale is a *difference scale*. The so-called Thurstone Case V scale, which is a measure of response strength, is an example of a difference scale.

2.2.2 Magnitude estimation, cross-modality matching, and interadjective comparison

Judgments of subjective loudness can be made in laboratory in various ways. Stevens (1957) classifies four methods. The method of *magnitude estimation* is one of the most common. The following provides an example:

Example 2 (Magnitude Estimation) *A subject hears a reference sound and is told to assign it a fixed number. Then he is presented other sounds and asked to assign them numbers proportional to the reference sound.*

Stevens argues that magnitude estimation gives rise to a *ratio scale*. Moreover, he uses the idea of *cross-modality matching* to test the *power law*. It must be noted that the scale corresponding to the power law is a *log-interval scale*. Krantz (1972) puts Stevens's argument that magnitude estimation gives rise to a ratio scale and his idea of cross-modality matching to test the power law on a rigorous *measurement-theoretic* foundation. In this paper, we try to propose a logic for *interadjective comparison* the model of which is based on Krantz's measurement theory for magnitude estimation and cross-modality matching.

2.2.3 Model

First, we define a model $\mathfrak{M}$ of $\mathscr{L}_{\mathsf{ICL}}$ as follows:

Definition 5 (Model) *$\mathfrak{M}$ is a sequence $(\mathscr{I}_{\mathrm{F_1}}, \ldots, \mathscr{I}_{\mathrm{F_n}}, a_1{}^{\mathfrak{M}}, b_1{}^{\mathfrak{M}}, \ldots, \spadesuit_{\mathrm{F_1}}, \ldots, \spadesuit_{\mathrm{F_n}}, \heartsuit_{\mathrm{F_1}}, \ldots, \heartsuit_{\mathrm{F_n}}, F_1{}^{\mathfrak{M}}, \ldots, F_{\mathrm{n}}{}^{\mathfrak{M}}, \succeq)$, where*

- *$\mathscr{I}_{\mathrm{F_i}}$ is a nonempty set of individuals for evaluation of F_{i}, celled a comparison class relative to F_{i}.*

- *$a_{\mathrm{i}}{}^{\mathfrak{M}}, b_{\mathrm{i}}{}^{\mathfrak{M}}, \ldots \in \mathscr{I}_{\mathrm{F_i}}$.*

- *$\spadesuit_{\mathrm{F_i}}$ is an average individual in $\mathscr{I}_{\mathrm{F_i}}$.*

- *$\heartsuit_{\mathrm{F_i}}$ is a zero-point individual in $\mathscr{I}_{\mathrm{F_i}}$.*

- *$F_{\mathrm{i}}{}^{\mathfrak{M}} \subseteq \mathscr{I}_{\mathrm{F_i}}$.*

- *$\succeq$ is a binary relation on $\bigcup_{\mathrm{i}=1}{}^{\mathrm{n}} \mathscr{I}_{\mathrm{F_i}} \times \mathscr{I}_{\mathrm{F_i}}$, called the interadjective-comparison ordering relation, that satisfies the following conditions:*

  1. *$\succeq$ is a weak order (transitive and connected).*

  2. *For any $a_{\mathrm{i}}{}^{\mathfrak{M}}, b_{\mathrm{i}}{}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_i}}$ and any $a_{\mathrm{j}}{}'^{\mathfrak{M}}, b_{\mathrm{j}}{}'^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_j}}$, if $(a_{\mathrm{i}}{}^{\mathfrak{M}}, b_{\mathrm{i}}{}^{\mathfrak{M}}) \succeq (a_{\mathrm{j}}{}'^{\mathfrak{M}}, b_{\mathrm{j}}{}'^{\mathfrak{M}})$, then $(b_{\mathrm{j}}{}'^{\mathfrak{M}}, a_{\mathrm{j}}{}'^{\mathfrak{M}}) \succeq (b_{\mathrm{i}}{}^{\mathfrak{M}}, a_{\mathrm{i}}{}^{\mathfrak{M}})$.*

  3. *For any $a_{\mathrm{i}}{}^{\mathfrak{M}}, b_{\mathrm{i}}{}^{\mathfrak{M}}, c_{\mathrm{i}}{}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_i}}$ and any $a_{\mathrm{j}}{}'^{\mathfrak{M}}, b_{\mathrm{j}}{}'^{\mathfrak{M}}, c_{\mathrm{j}}{}'^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_j}}$, if $(a_{\mathrm{i}}{}^{\mathfrak{M}}, b_{\mathrm{i}}{}^{\mathfrak{M}}) \succeq (a_{\mathrm{j}}{}'^{\mathfrak{M}}, b_{\mathrm{j}}{}'^{\mathfrak{M}})$ and $(b_{\mathrm{i}}{}^{\mathfrak{M}}, c_{\mathrm{i}}{}^{\mathfrak{M}}) \succeq (b_{\mathrm{j}}{}'^{\mathfrak{M}}, c_{\mathrm{j}}{}'^{\mathfrak{M}})$, then $(a_{\mathrm{i}}{}^{\mathfrak{M}}, c_{\mathrm{i}}{}^{\mathfrak{M}}) \succeq (a_{\mathrm{j}}{}'^{\mathfrak{M}}, c_{\mathrm{j}}{}'^{\mathfrak{M}})$.*

  4. *For any $a_{\mathrm{i}}{}^{\mathfrak{M}}, b_{\mathrm{i}}{}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_i}}$, there exist $a_1{}'^{\mathfrak{M}}, b_1{}'^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_1}}$ such that $(a_{\mathrm{i}}{}^{\mathfrak{M}}, b_{\mathrm{i}}{}^{\mathfrak{M}}) \sim (a_1{}'^{\mathfrak{M}}, b_1{}'^{\mathfrak{M}})$, where $(a_{\mathrm{i}}{}^{\mathfrak{M}}, b_{\mathrm{i}}{}^{\mathfrak{M}}) \sim (a_{\mathrm{j}}{}'^{\mathfrak{M}}, b_{\mathrm{j}}{}'^{\mathfrak{M}}) := (a_{\mathrm{i}}{}^{\mathfrak{M}}, b_{\mathrm{i}}{}^{\mathfrak{M}}) \succeq (a_{\mathrm{j}}{}'^{\mathfrak{M}}, b_{\mathrm{j}}{}'^{\mathfrak{M}})$ and $(a_{\mathrm{j}}{}'^{\mathfrak{M}}, b_{\mathrm{j}}{}'^{\mathfrak{M}}) \succeq (a_{\mathrm{i}}{}^{\mathfrak{M}}, b_{\mathrm{i}}{}^{\mathfrak{M}})$.*

  5. *For any $a_1{}^{\mathfrak{M}}, b_1{}^{\mathfrak{M}}, c_1{}^{\mathfrak{M}}, d_1{}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_1}}$, if $(d_1{}^{\mathfrak{M}}, c_1{}^{\mathfrak{M}}) \succeq (a_1{}^{\mathfrak{M}}, b_1{}^{\mathfrak{M}}) \succeq (d_1{}^{\mathfrak{M}}, d_1{}^{\mathfrak{M}})$, then there exist $a_1{}'^{\mathfrak{M}}, b_1{}'^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_1}}$ such that $(d_1{}^{\mathfrak{M}}, b_1{}'^{\mathfrak{M}}) \sim (a_1{}^{\mathfrak{M}}, b_1{}^{\mathfrak{M}}) \sim (a_1{}'^{\mathfrak{M}}, c_1{}^{\mathfrak{M}})$.*

  6. *Suppose that $a_1{}^{(1)\mathfrak{M}}, a_1{}^{(2)\mathfrak{M}}, \ldots, a_1{}^{(\mathrm{i})\mathfrak{M}}, \ldots$ is a sequence of equally spaced elements of $\mathscr{I}_{\mathrm{F_1}}$, that is, $(a_1{}^{(\mathrm{i}+1)\mathfrak{M}}, a_1{}^{(\mathrm{i})\mathfrak{M}}) \sim (a_1{}^{(2)\mathfrak{M}}, a_1{}^{(1)\mathfrak{M}}) \succ (a_1{}^{(1)\mathfrak{M}}, a_1{}^{(1)\mathfrak{M}})$ for any*

$a_1{}^{(\mathrm{i+1})\mathfrak{M}}, a_1{}^{(\mathrm{i})\mathfrak{M}}$ *in the sequence. If the sequence is strictly bounded (that is, if there exist* $b_1{}^{\mathfrak{M}}, c_1{}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F}_1}$ *such that* $(b_1{}^{\mathfrak{M}}, c_1{}^{\mathfrak{M}}) \succ (a_1{}^{(\mathrm{i})\mathfrak{M}}, a_1{}^{(1)\mathfrak{M}})$ *for any* $a_1{}^{(\mathrm{i})\mathfrak{M}}$ *in the sequence), then it is finite.*

Condition 2 postulates that reversing pairs should be reversing the ordering. The following provides an example:

Example 3 (Reversal of Pairs) *In 1(b), if* a_i *is much taller than* b_i*, and* a_j' *is slightly more intelligent than* b_j'*, so that* $(a_\mathrm{i}{}^{\mathfrak{M}}, b_\mathrm{i}{}^{\mathfrak{M}}) \succeq (a_\mathrm{j}{}^{\prime\mathfrak{M}}, b_\mathrm{j}{}^{\prime\mathfrak{M}})$*, then* b_i *is much shorter than* a_i*, but* b_j' *is only slightly duller than* a_j'*, so that* $(b_\mathrm{j}{}^{\prime\mathfrak{M}}, a_\mathrm{j}{}^{\prime\mathfrak{M}}) \succeq (b_\mathrm{i}{}^{\mathfrak{M}}, a_\mathrm{i}{}^{\mathfrak{M}})$.

Condition 3 says that pairs $(a_\mathrm{i}{}^{\mathfrak{M}}, b_\mathrm{i}{}^{\mathfrak{M}})$ behave *qualitatively* like ratios or differences with respect to $\succeq$. The following provides an example:

Example 4 (Ratios and Differences) *Pairs* $(a_\mathrm{i}{}^{\mathfrak{M}}, b_\mathrm{i}{}^{\mathfrak{M}})$ *behave with respect to* $\succeq$ *in much the same as* $\frac{a_\mathrm{i}}{b_\mathrm{i}} \geq \frac{a_\mathrm{j}'}{b_\mathrm{j}'}$ *and* $\frac{b_\mathrm{i}}{c_\mathrm{i}} \geq \frac{b_\mathrm{j}'}{c_\mathrm{i}'}$ *implies that* $\frac{a_\mathrm{i}}{c_\mathrm{i}} \geq \frac{a_\mathrm{j}'}{c_\mathrm{j}'}$ *(for positive real numbers) and* $a_\mathrm{i} - b_\mathrm{i} \geq a_\mathrm{j}' - b_\mathrm{j}'$ *and* $b_\mathrm{i} - c_\mathrm{i} \geq b_\mathrm{j}' - c_\mathrm{j}'$ *implies* $a_\mathrm{i} - c_\mathrm{i} \geq a_\mathrm{j}' - c_\mathrm{j}'$.

Condition 4 postulates that any pair $(a_\mathrm{i}{}^{\mathfrak{M}}, b_\mathrm{i}{}^{\mathfrak{M}})$ should be equivalent to some $\mathscr{I}_{\mathrm{F}_1} \times \mathscr{I}_{\mathrm{F}_1}$ pair. Condition 5 postulates that intermediate-level individuals can be chosen *densely* within $\mathscr{I}_{\mathrm{F}_1}$. Condition 6 postulates that $\succeq$ should have an *Archimedean Property*. The following provides an example:

Example 5 (Archimedean Property) *However small the pair* $(a_1{}^{\mathfrak{M}}, b_1{}^{\mathfrak{M}})$ *in tallness, if one can find a sequence of tallness* $a_1{}^{(1)\mathfrak{M}}, a_1{}^{(2)\mathfrak{M}}, \ldots$ *such that any pair* $(a_1{}^{(\mathrm{i+1})\mathfrak{M}}, a_1{}^{(\mathrm{i})\mathfrak{M}})$ *in tallness is equivalent to* $(a_1{}^{\mathfrak{M}}, b_1{}^{\mathfrak{M}})$ *in tallness, then the overall interval* $(a_1{}^{(\mathrm{n})\mathfrak{M}}, a_1{}^{(1)\mathfrak{M}})$ *in tallness becomes indefinitely large.*

We can prove the following representation theorem for interadjective-comparison ordering by modifying the method of (Krantz et al., 1971):

Theorem 1 (Representation for Interadjective-Comparison Ordering) *If* $\succeq$ *is an interadjective-comparison ordering relation of Definition 5, then there exist functions* $f_\mathrm{i} : \mathscr{I}_\mathrm{i} \to \mathbb{R}^+$ $(1 \leq i \leq n)$ *such that for any* $a_\mathrm{i}{}^{\mathfrak{M}}, b_\mathrm{i}{}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F}_\mathrm{i}}$ $(1 \leq i \leq n)$ *and any* $a_\mathrm{j}{}^{\prime\mathfrak{M}}, b_\mathrm{j}{}^{\prime\mathfrak{M}} \in \mathscr{I}_{\mathrm{F}_\mathrm{j}}$ $(1 \leq j \leq n)$,

$$2(e) \quad (a_\mathrm{i}{}^{\mathfrak{M}}, b_\mathrm{i}{}^{\mathfrak{M}}) \succeq (a_\mathrm{j}{}^{\prime\mathfrak{M}}, b_\mathrm{j}{}^{\prime\mathfrak{M}}) \text{ iff } \frac{f_\mathrm{i}(a_\mathrm{i}{}^{\mathfrak{M}})}{f_\mathrm{i}(b_\mathrm{i}{}^{\mathfrak{M}})} \geq \frac{f_\mathrm{j}(a_\mathrm{j}{}^{\prime\mathfrak{M}})}{f_\mathrm{j}(b_\mathrm{j}{}^{\prime\mathfrak{M}})}.$$

We can also prove the following uniqueness theorem for interadjective-comparison ordering by modifying the method of (Krantz et al., 1971):

Theorem 2 (Uniqueness for Interadjective-Comparison Ordering) *If* f'_i $(1 \leq i \leq n)$ *are any other such functions as* f_i *of Theorem 1, then there exist* $\alpha_\mathrm{i}, \beta \in \mathbb{R}^+$ $(1 \leq i \leq n)$ *such that*

$$f'_\mathrm{i} = \alpha_\mathrm{i} f_\mathrm{i}{}^{\beta}.$$

Remark 1 f_i *defines a log-interval scale.*

2.2.4 Magnitude estimation and ratio scale

We specify some conditions under which *magnitude estimation* leads to a *ratio scale*[5]. These conditions follow from the following three consistency conditions:

1. the magnitude-pair consistency condition,
2. the pair consistency condition, and
3. the magnitude-interadjective-comparison consistency condition.

We state these three consistency conditions. Suppose that an experimenter is performing a magnitude estimation of a subject on $\mathscr{I}_{\mathrm{F_i}}$. The experimenter fixes $a_{\mathrm{i}}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_i}}$ and assigns to $a_{\mathrm{i}}^{\mathfrak{M}}$ the psychological magnitude p. We assume that all magnitudes are positive. Next, the experimenter asks the subject to assign to each $b_{\mathrm{i}}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_i}}$ a magnitude q depending on $a_{\mathrm{i}}^{\mathfrak{M}}$ and p. This is written in symbols as follows:

$$ME_{\mathrm{i}}(b_{\mathrm{i}}^{\mathfrak{M}}|a_{\mathrm{i}}^{\mathfrak{M}}, p) = q,$$

where ME_{i} is the magnitude estimate for $b_{\mathrm{i}}^{\mathfrak{M}}$ when $a_{\mathrm{i}}^{\mathfrak{M}}$ is assigned a magnitude p. In particular,

$$ME_{\mathrm{i}}(a_{\mathrm{i}}^{\mathfrak{M}}|a_{\mathrm{i}}^{\mathfrak{M}}, p) = p.$$

In the variant of magnitude estimation called *pair estimation*, a pair of $a_{\mathrm{i}}^{\mathfrak{M}}$ and $b_{\mathrm{i}}^{\mathfrak{M}}$ from $\mathscr{I}_{\mathrm{F_i}}$ are presented, and then the experimenter asks the subject to give a numerical estimate of, as it were, the qualitative ratio of $a_{\mathrm{i}}^{\mathfrak{M}}$ to $b_{\mathrm{i}}^{\mathfrak{M}}$. We denote this estimate by $PE_{\mathrm{i}}(a_{\mathrm{i}}^{\mathfrak{M}}, b_{\mathrm{i}}^{\mathfrak{M}})$. It is reasonable to assume that PE_{i} *corresponds to* $\succeq$ as follows:

$$2(f) \quad (a_{\mathrm{i}}^{\mathfrak{M}}, b_{\mathrm{i}}^{\mathfrak{M}}) \succeq (c_{\mathrm{j}}^{\mathfrak{M}}, d_{\mathrm{j}}^{\mathfrak{M}}) \text{ iff } PE_{\mathrm{i}}(a_{\mathrm{i}}^{\mathfrak{M}}, b_{\mathrm{i}}^{\mathfrak{M}}) \geq PE_{\mathrm{j}}(c_{\mathrm{j}}^{\mathfrak{M}}, d_{\mathrm{j}}^{\mathfrak{M}}).$$

Magnitude estimates and pair estimates are often assumed to satisfy the following *magnitude-pair consistency condition*: for any $c_{\mathrm{i}}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{i}}$ and any $p \in \mathbb{R}^+$,

$$2(g) \quad PE_{\mathrm{i}}(a_{\mathrm{i}}^{\mathfrak{M}}, b_{\mathrm{i}}^{\mathfrak{M}}) = \frac{ME_{\mathrm{i}}(a_{\mathrm{i}}^{\mathfrak{M}}|c_{\mathrm{i}}^{\mathfrak{M}}, p)}{ME_{\mathrm{i}}(b_{\mathrm{i}}^{\mathfrak{M}}|c_{\mathrm{i}}^{\mathfrak{M}}, p)}.$$

Moreover, it is often assumed that pair estimates behave like rations, that is, they satisfy the following *pair consistency condition*: for any $a_{\mathrm{i}}^{\mathfrak{M}}, b_{\mathrm{i}}^{\mathfrak{M}}, c_{\mathrm{i}}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{i}}$,

$$2(h) \quad PE_{\mathrm{i}}(a_{\mathrm{i}}^{\mathfrak{M}}, b_{\mathrm{i}}^{\mathfrak{M}}) \cdot PE_{\mathrm{i}}(b_{\mathrm{i}}^{\mathfrak{M}}, c_{\mathrm{i}}^{\mathfrak{M}}) = PE_{\mathrm{i}}(a_{\mathrm{i}}^{\mathfrak{M}}, c_{\mathrm{i}}^{\mathfrak{M}}).$$

If $i = 1$, 2(h) yields

$$2(i) \quad PE_1(a_1^{\mathfrak{M}}, b_1^{\mathfrak{M}}) \cdot PE_1(b_1^{\mathfrak{M}}, c_1^{\mathfrak{M}}) = PE_1(a_1^{\mathfrak{M}}, c_1^{\mathfrak{M}}).$$

In *interadjective-comparison matching*, we usually fix $a_{\mathrm{i}}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_i}}$ and $a_{\mathrm{j}}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_j}}$ and say that they match. The experimenter then asks the subject to find $b_{\mathrm{i}}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_i}}$ that matches a given individual $b_{\mathrm{j}}^{\mathfrak{M}} \in \mathscr{I}_{\mathrm{F_j}}$. This is written in symbols as follows:

$$IM_{\mathrm{ji}}(b_{\mathrm{j}}^{\mathfrak{M}}|a_{\mathrm{j}}^{\mathfrak{M}}, a_{\mathrm{i}}^{\mathfrak{M}}) = b_{\mathrm{i}}^{\mathfrak{M}}.$$

[5]We owe this subsection to (Roberts, 1979:pp.186–192)

In particular,

$$IM_{\mathrm{ji}}({a_{\mathrm{j}}}^{\mathfrak{M}}|{a_{\mathrm{j}}}^{\mathfrak{M}},{a_{\mathrm{i}}}^{\mathfrak{M}}) = {a_{\mathrm{i}}}^{\mathfrak{M}}.$$

It is reasonable to assume that *if* ${a_{\mathrm{j}}}^{\mathfrak{M}}$ *is matched by* ${a_{\mathrm{i}}}^{\mathfrak{M}}$ *and* ${b_{\mathrm{j}}}^{\mathfrak{M}}$ *by* ${b_{\mathrm{i}}}^{\mathfrak{M}}$*, then the corresponding qualitative ratios are judged equal*:

$$2(j) \quad \text{If } IM_{\mathrm{ji}}({b_{\mathrm{j}}}^{\mathfrak{M}}|{a_{\mathrm{j}}}^{\mathfrak{M}},{a_{\mathrm{i}}}^{\mathfrak{M}}) = {b_{\mathrm{i}}}^{\mathfrak{M}}, \text{ then } ({b_{\mathrm{j}}}^{\mathfrak{M}},{a_{\mathrm{j}}}^{\mathfrak{M}}) \sim ({b_{\mathrm{i}}}^{\mathfrak{M}},{a_{\mathrm{i}}}^{\mathfrak{M}}).$$

It is often assumed that magnitude estimation and interadjective-comparison matching are related by the following *magnitude-interadjective-comparison consistency condition*:

$$2(k) \quad \frac{ME_{\mathrm{i}}(IM_{\mathrm{ji}}({b_{\mathrm{j}}}^{\mathfrak{M}}|{a_{\mathrm{j}}}^{\mathfrak{M}},{a_{\mathrm{i}}}^{\mathfrak{M}})|{c_{\mathrm{i}}}^{\mathfrak{M}},q)}{ME_{\mathrm{i}}({a_{\mathrm{i}}}^{\mathfrak{M}}|{c_{\mathrm{i}}}^{\mathfrak{M}},p)} = \frac{ME_{\mathrm{j}}({b_{\mathrm{j}}}^{\mathfrak{M}}|{c_{\mathrm{j}}}^{\mathfrak{M}},q)}{ME_{\mathrm{j}}({a_{\mathrm{j}}}^{\mathfrak{M}}|{c_{\mathrm{j}}}^{\mathfrak{M}},q)}.$$

That is, if ${b_{\mathrm{j}}}^{\mathfrak{M}}$ is matched with ${b_{\mathrm{i}}}^{\mathfrak{M}}$ in the interadjective-comparison matching, where ${a_{\mathrm{j}}}^{\mathfrak{M}}$ is given as matched with ${a_{\mathrm{i}}}^{\mathfrak{M}}$, then the ratio of the magnitude estimate of ${b_{\mathrm{i}}}^{\mathfrak{M}}$ to the magnitude estimate of ${a_{\mathrm{i}}}^{\mathfrak{M}}$ on the ith adjective equals the ratio of the magnitude of ${b_{\mathrm{j}}}^{\mathfrak{M}}$ to the magnitude estimate of ${a_{\mathrm{j}}}^{\mathfrak{M}}$ on the jth adjective for any reference estimate p for ${c_{\mathrm{i}}}^{\mathfrak{M}}$ and q for ${c_{\mathrm{j}}}^{\mathfrak{M}}$. If $IM_{\mathrm{ji}}({b_{\mathrm{j}}}^{\mathfrak{M}}|{a_{\mathrm{j}}}^{\mathfrak{M}},{a_{\mathrm{i}}}^{\mathfrak{M}}) = {b_{\mathrm{i}}}^{\mathfrak{M}}$ and ${a_{\mathrm{i}}}^{\mathfrak{M}} = {c_{\mathrm{i}}}^{\mathfrak{M}}$, 2 (g) and 2 (k) yield

$$2(l) \quad \frac{ME_{\mathrm{i}}({b_{\mathrm{i}}}^{\mathfrak{M}}|{a_{\mathrm{i}}}^{\mathfrak{M}},p)}{p} = PE_{\mathrm{j}}({b_{\mathrm{j}}}^{\mathfrak{M}},{a_{\mathrm{j}}}^{\mathfrak{M}})$$

because $ME_{\mathrm{i}}({a_{\mathrm{i}}}^{\mathfrak{M}}|{a_{\mathrm{i}}}^{\mathfrak{M}},p) = p$. It is reasonable to assume that if $({b_{\mathrm{i}}}^{\mathfrak{M}},{a_{\mathrm{i}}}^{\mathfrak{M}}) \sim ({b_1}^{\mathfrak{M}},{a_1}^{\mathfrak{M}})$, then 2(l) holds for $j = 1$:

$$2(m) \quad \text{If } ({b_{\mathrm{i}}}^{\mathfrak{M}},{a_{\mathrm{i}}}^{\mathfrak{M}}) \sim ({b_1}^{\mathfrak{M}},{a_1}^{\mathfrak{M}}), \text{ then } \frac{ME_{\mathrm{i}}({b_{\mathrm{i}}}^{\mathfrak{M}}|{a_{\mathrm{i}}}^{\mathfrak{M}},p)}{p} = PE_1({b_1}^{\mathfrak{M}},{a_1}^{\mathfrak{M}}).$$

We can prove the following representation theorem for magnitude estimation by modifying the method of (Krantz, 1972):

Theorem 3 (Representation for Magnitude Estimation) *Suppose that* $\succeq$ *is an interadjective-comparison ordering relation of Definition 5. Moreover, suppose that* $\succeq, ME_{\mathrm{i}}, PE_{\mathrm{i}}$ *and* IM_{i} *satisfy 2(f), 2(i), 2(j) and 2(m). Then there exists a power function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that if* f_{i}*s satisfy 2 (e), then*

$$2(n) \quad ME_{\mathrm{i}}({b_{\mathrm{i}}}^{\mathfrak{M}}|{a_{\mathrm{i}}}^{\mathfrak{M}},p) = q \text{ iff } \frac{f_{\mathrm{i}}({b_{\mathrm{i}}}^{\mathfrak{M}})}{f_{\mathrm{i}}({a_{\mathrm{i}}}^{\mathfrak{M}})} = \frac{f(q)}{f(p)},$$

$$2(o) \quad PE_{\mathrm{i}}({a_{\mathrm{i}}}^{\mathfrak{M}},{b_{\mathrm{i}}}^{\mathfrak{M}}) = r \text{ iff } \frac{f_{\mathrm{i}}({a_{\mathrm{i}}}^{\mathfrak{M}})}{f_{\mathrm{i}}({b_{\mathrm{i}}}^{\mathfrak{M}})} = f(r), \text{ and}$$

$$2(p) \quad \text{If } IM_{\mathrm{ji}}({b_{\mathrm{j}}}^{\mathfrak{M}}|{a_{\mathrm{j}}}^{\mathfrak{M}},{a_{\mathrm{i}}}^{\mathfrak{M}}) = {b_{\mathrm{i}}}^{\mathfrak{M}}, \text{ then } \frac{f_{\mathrm{i}}({b_{\mathrm{i}}}^{\mathfrak{M}})}{f_{\mathrm{j}}({b_{\mathrm{j}}}^{\mathfrak{M}})} = \frac{f_{\mathrm{i}}({a_{\mathrm{i}}}^{\mathfrak{M}})}{f_{\mathrm{j}}({a_{\mathrm{j}}}^{\mathfrak{M}})}.$$

We can also prove the following uniqueness theorem for magnitude estimation by modifying the method of (Krantz, 1972):

Theorem 4 (Uniqueness for Magnitude Estimation) *If f'_{i} and f' also satisfy 2 (e) and 2 (n) through 2(p), then there exist there exist $\alpha_{\mathrm{i}}, \beta \in \mathbb{R}^+$ $(1 \leq i \leq n)$ such that*

$$f'_{\mathrm{i}} = \alpha_{\mathrm{i}} {f_{\mathrm{i}}}^{\beta} \text{ and } f' = f^{\beta}.$$

Remark 2 *Both f_{i} and f define log-interval scales.*

We now obtain the following corollary of Theorem 3 and Theorem 4:

Corollary 1 (Ratio Scale) *ME_{i} defines a ratio scale.*

2.2.5 Satisfaction, truth and validity

We define an assignment function and its extension as follows:

Definition 6 (Assignment Function and Its Extension) *Let $\mathscr{V}$ denote a set of individual variables, $\mathscr{C}$ a set of individual constants and $\mathscr{I}$ a set of individuals.*

- *We call $s : \mathscr{V} \to \mathscr{I}$ an assignment function.*
- *We define the extension of s as a function $\tilde{s} : \mathscr{V} \cup \mathscr{C} \to \mathscr{I}$ such that*
  1. *For any $x \in \mathscr{V}$, $\tilde{s}(x) = s(x)$, and*
  2. *For any $a \in \mathscr{C}$, $\tilde{s}(a) = a^{\mathfrak{M}}$.*

We provide ICL with the following satisfaction definition relative to $\mathfrak{M}$, define the truth in $\mathfrak{M}$ by means of satisfaction, and then define validity as follows:

Definition 7 (Satisfaction, Truth and Validity) *What it means for $\mathfrak{M}$ to satisfy $\varphi \in \Phi_{\mathscr{L}_{\mathsf{ICL}}}$ with s, in symbols $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \varphi[s]$ is inductively defined as follows:*

- $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} P(t)[s]$ *iff* $\tilde{s}(t) \in P^{\mathfrak{M}}$.
- $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} t_1 = t_2[s]$ *iff* $\tilde{s}(t_1) = \tilde{s}(t_2)$.
- *2(q)* $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{DER}_{\mathrm{P_i,P_j}}(t_{\mathrm{i}}, t_{\mathrm{j}})[s]$ *iff if $\succeq, ME_{\mathrm{i}}, PE_{\mathrm{i}}$ and IM_{i} satisfy 2(f), 2(i), 2(j) and 2(m), then $\tilde{s}(t_{\mathrm{i}}) \in \mathscr{I}_{\mathrm{P_i}}$ and $\tilde{s}(t_{\mathrm{j}}) \in \mathscr{I}_{\mathrm{P_j}}$ and $ME_{\mathrm{i}}(\tilde{s}(t_{\mathrm{i}})|\heartsuit_{\mathrm{P_i}}, 0) > ME_{\mathrm{j}}(\tilde{s}(t_{\mathrm{j}})|\heartsuit_{\mathrm{P_j}}, 0)$.*
- *2(r)* $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{IER}_{\mathrm{P_i,P_j}}(t_{\mathrm{i}}, t_{\mathrm{j}})[s]$ *iff $\tilde{s}(t_{\mathrm{i}}) \in \mathscr{I}_{\mathrm{P_i}}$ and $\tilde{s}(t_{\mathrm{j}}) \in \mathscr{I}_{\mathrm{P_j}}$ and $(\tilde{s}(t_{\mathrm{i}}), \spadesuit_{\mathrm{i}}) \succ (\tilde{s}(t_{\mathrm{j}}), \spadesuit_{\mathrm{j}})$, where $(\tilde{s}(t_{\mathrm{i}}), \tilde{s}(t_{\mathrm{j}})) \succ (\tilde{s}(t_{\mathrm{k}}), \tilde{s}(t_{\mathrm{l}})) := (\tilde{s}(t_{\mathrm{k}}), \tilde{s}(t_{\mathrm{l}})) \not\succeq (\tilde{s}(t_{\mathrm{i}}), \tilde{s}(t_{\mathrm{j}}))$.*
- *2(s)* $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{TER}_{\mathrm{P_i,P_j}}{}^{\mathrm{k}}(t_{\mathrm{i}}, t_{\mathrm{j}})[s]$ *iff if $\succeq, ME_{\mathrm{i}}, PE_{\mathrm{i}}$ and IM_{i} satisfy 2(f), 2(i), 2(j) and 2(m), then $\tilde{s}(t_{\mathrm{i}}) \in \mathscr{I}_{\mathrm{P_i}}$ and $\tilde{s}(t_{\mathrm{j}}) \in \mathscr{I}_{\mathrm{P_j}}$ and $ME_{\mathrm{i}}(\tilde{s}(t_{\mathrm{i}})|\heartsuit_{\mathrm{P_i}}, 0) = k \cdot ME_{\mathrm{j}}(\tilde{s}(t_{\mathrm{j}})|\heartsuit_{\mathrm{P_j}}, 0)$.*
- *2(t)* $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{UER}_{\mathrm{P_i,P_j}}{}^{\mathrm{k}}(t_{\mathrm{i}}, t_{\mathrm{j}})[s]$ *iff if $\succeq, ME_{\mathrm{i}}, PE_{\mathrm{i}}$ and IM_{i} satisfy 2(f), 2(i), 2(j) and 2(m), then $\tilde{s}(t_{\mathrm{i}}) \in \mathscr{I}_{\mathrm{P_i}}$ and $\tilde{s}(t_{\mathrm{j}}) \in \mathscr{I}_{\mathrm{P_j}}$ and $ME_{\mathrm{i}}(\tilde{s}(t_{\mathrm{i}})|\heartsuit_{\mathrm{P_i}}, 0) = ME_{\mathrm{j}}(\tilde{s}(t_{\mathrm{j}})|\heartsuit_{\mathrm{P_j}}, 0) + k$.*

- *2(u)* $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{FDER}_{\mathrm{P_i,P_j}}(t_\mathrm{i},t_\mathrm{j},t_\mathrm{l},t_\mathrm{m})[s]$ *iff if* $\succeq, ME_\mathrm{i}, PE_\mathrm{i}$ *and* IM_i *satisfy 2(f), 2(i), 2(j) and 2(m), then* $\tilde{s}(t_\mathrm{i}),\tilde{s}(t_\mathrm{j}) \in \mathscr{I}_{\mathrm{P_i}}$ *and* $\tilde{s}(t_\mathrm{l}),\tilde{s}(t_\mathrm{m}) \in \mathscr{I}_{\mathrm{P_j}}$ *and* $\dfrac{ME_\mathrm{i}(\tilde{s}(t_\mathrm{i})|\tilde{s}(t_\mathrm{j}),p)}{p} > \dfrac{ME_\mathrm{j}(\tilde{s}(t_\mathrm{l})|\tilde{s}(t_\mathrm{m}),q)}{q}$.

- *2(v)* $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{FIER}_{\mathrm{P_i,P_j}}(t_\mathrm{i},t_\mathrm{j},t_\mathrm{l},t_\mathrm{m})[s]$ *iff* $\tilde{s}(t_\mathrm{i}),\tilde{s}(t_\mathrm{j}) \in \mathscr{I}_{\mathrm{P_i}}$ *and* $\tilde{s}(t_\mathrm{l}),\tilde{s}(t_\mathrm{m}) \in \mathscr{I}_{\mathrm{P_j}}$ *and* $(t_\mathrm{i}^{\mathfrak{M}},\tilde{s}(t_\mathrm{j})) \succ (\tilde{s}(t_\mathrm{l}),\tilde{s}(t_\mathrm{m}))$.

- $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \top$.

- $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \neg\varphi[s]$ *iff* $\mathfrak{M} \not\models_{\mathscr{L}_{\mathsf{ICL}}} \varphi[s]$.

- $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \varphi \wedge \psi[s]$ *iff* $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \varphi[s]$ *and* $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \psi[s]$.

- $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \forall x\varphi[s]$ *iff for any* $\mathfrak{d} \in \mathscr{I}$, $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \varphi[s(x|\mathfrak{d})]$,
where $s(x|\mathfrak{d})$ *is the function that is exactly like s except for one thing: for the individual variable x, it assigns the individual* $\mathfrak{d}$. *This can be expressed as follows:*

$$s(x|\mathfrak{d})(y) := \begin{cases} s(y) & \text{if} \quad y \neq x \\ \mathfrak{d} & \text{if} \quad y = x. \end{cases}$$

If $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \varphi[s]$ *for all s, we write* $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \varphi$ *and say that* φ *is true in* $\mathfrak{M}$. *If* φ *is true in all models of* $\mathscr{L}_{\mathsf{ICL}}$, *we write* $\models_{\mathscr{L}_{\mathsf{ICL}}} \varphi$ *and say that* φ *is valid.*

2.3 Non First-Order axiomatisability of models of $\mathscr{L}_{\mathsf{ICL}}$

The semantic structure of ICL is so *rich* that ICL has the following meta-logical property.

Theorem 5 (Non First-Order Axiomatisability) *The class of models of* $\mathscr{L}_{\mathsf{ICL}}$ *is not first-order axiomatisable.*

Remark 3 *We can express the Archimedean Property by means of infinite quantifier sequences. In order to express them, we need infinitary logic.*

2.4 Truth conditions of examples

When a model $\mathfrak{M}$ of $\mathscr{L}_{\mathsf{ICL}}$ is given, the truth conditions of the examples 1(a), 1(b), 1(d), 1(e), 1(f), and 1(g) are as follows:

- 1(a) Albert is taller than Charles is wide.
$\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{DER}_{\mathrm{tall,wide}}(Albert, Charles)$ iff if $\succeq, ME_1, ME_2, PE_1, PE_2, IM_1$ and IM_2 satisfy 2(f), 2(g), 2(i), 2(j) and 2(m), then $Albert \in \mathscr{I}_{\mathrm{tall}}$ and $Charles \in \mathscr{I}_{\mathrm{wide}}$ and $ME_1(Albert|\heartsuit_{\mathrm{tall}},0) > ME_2(Charles|\heartsuit_{\mathrm{wide}},0)$ ($\because 2(q)$).

- 1(b) Albert is taller than Charles is intelligent.
$\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{IER}_{\text{tall,intelligent}}(Albert, Charles)$ iff $Albert \in \mathscr{I}_{\text{tall}}$ and $Charles \in \mathscr{I}_{\text{intelligent}}$ and $(Albert, \spadesuit_{\text{tall}}) \succ (Charles, \spadesuit_{\text{intelligent}})$ ($\because$ 2(r)).

- 1(d) Albert is three times as tall as Charles is wide.
$\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{TER}_{\text{tall,wide}}{}^{3}(Albert, Charles)[s]$ iff if $\succeq, ME_1, ME_2, PE_1, PE_2, IM_1$ and IM_2 satisfy 2(f), 2(g), 2(i), 2(j) and 2(m), then $Albert \in \mathscr{I}_{\text{tall}}$ and $Charles \in \mathscr{I}_{\text{wide}}$ and $ME_1(Albert|\heartsuit_{\text{tall}}, 0) = 3 \cdot ME_2(Charles|\heartsuit_{\text{wide}}, 0)$ ($\because$ 2(s)).

- 1(e) Albert is 10 centimetres taller than Charles is wide.
$\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{TER}_{\text{tall,wide}}{}^{10}(Albert, Charles)$ iff if $\succeq, ME_1, ME_2, PE_1, PE_2, IM_1$ and IM_2 satisfy 2(f), 2(i), 2(j) and 2(m), then $Albert \in \mathscr{I}_{\text{tall}}$ and $Charles \in \mathscr{I}_{\text{wide}}$ and $ME_1(Albert|\heartsuit_{\text{tall}}, 0) = ME_2(Charles|\heartsuit_{\text{wide}}, 0) + 10$ ($\because$ 2(t)).

- 1(f) Albert is taller than Bernard by more than Charles is wider than Dennis.
$\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{FDER}_{\text{tall,wide}}(Albert, Bernard, Charles, Dennis)$ iff if $\succeq, ME_1, ME_2, PE_1, PE_2, IM_1$ and IM_2 satisfy 2(f), 2(i), 2(j) and 2(m), then $Albert, Bernard \in \mathscr{I}_{\text{tall}}$ and $Charles, Dennis \in \mathscr{I}_{\text{wide}}$ and $\dfrac{ME_1(Albert|Bernard, p)}{p} > \dfrac{ME_2(Charles|Dennis, q)}{q}$ ($\because$ 2(u)).

- 1(g) Albert is taller than Bernard by more than Charles is more intelligent than Dennis.
$\mathfrak{M} \models_{\mathscr{L}_{\mathsf{ICL}}} \mathbf{FIER}_{\text{tall,wide}}(Albert, Bernard, Charles, Dennis)$ iff $Albert, Bernard \in \mathscr{I}_{\text{tall}}$ and $Charles, Dennis \in \mathscr{I}_{\text{wide}}$ and $(Albert, Bernard) \succ (Charles, Dennis)$ ($\because$ 2(v)).

2.5 Why is 1(c) meaningless?

Let us now return to the question as to why 1(c) is meaningless:

1(c) Albert is five times as tall as Charles is intelligent.

In order to provide 1(c) with its truth condition in terms of 2(s), PE_1 and ME_2 must satisfy the instance of 2(m):

$$2(w)\quad \begin{array}{l} \text{If } ({b_2}^{\mathfrak{M}}, {a_2}^{\mathfrak{M}}) \sim ({b_1}^{\mathfrak{M}}, {a_1}^{\mathfrak{M}}), \text{ then } \dfrac{ME_2({b_2}^{\mathfrak{M}}|{a_2}^{\mathfrak{M}}, p)}{p} = PE_1({b_1}^{\mathfrak{M}}, {a_1}^{\mathfrak{M}}), \\ \text{for any } {a_1}^{\mathfrak{M}}, {b_1}^{\mathfrak{M}} \in \mathscr{I}_{\text{tall}} \text{ and any } {a_2}^{\mathfrak{M}}, {b_2}^{\mathfrak{M}} \in \mathscr{I}_{\text{intelligent}}. \end{array}$$

However, in the semantic structure of 1(c), we cannot construct PE_1 and ME_2 that satisfy 2(w). Therefore, we cannot provide 1(c) with its truth condition in terms of 2(s).

3 Concluding remarks

In this paper, we have proposed a new version of logic for interadjective comparisons–Interadjective-Comparison Logic (ICL). In terms of measurement theory, the model of the language of ICL can provide reasonable conditions for log-interval scales and ratio scales, give the truth conditions of interadjective comparatives like 1(a), 1(b), 1(d), 1(e), 1(f) and 1(g), and render interadjective comparatives like 1(c) meaningless.

This paper is only a part of a larger measurement-theoretic study. We are now trying to construct such logics as

1. dynamic epistemic preference logic (Suzuki, 2009b),
2. dyadic deontic logic (Suzuki, 2009a),
3. vague predicate logic (Suzuki, 2011c,d),
4. threshold utility maximiser's preference logic (Suzuki, 2011a), and
5. a logic of questions and answers

by means of measurement theory.

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