The scope of supplements[1]

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Abstract. This paper defends the bidimensional approach (Potts 2005) to the semantics of supplements against criticisms based on non-projecting supplements. We discuss the empirical and theoretical aspects of two opposing representatives, Martin (2017) and Schlenker (2023), and propose a bidimensional semantics where supplements take scope when needed. The proposals resolves the issues arising from non-projecting supplements while retaining the explanatory advantages of bidimensionalism.

Keywords: non-restrictive relative clauses, supplements, projection, bidimensionalism, scope, monad

1. Introduction

Supplemental contents, or supplements, such as those introduced by **non-restrictive relative clauses (NRRCs)** (as highlighted in italics below), present a fascinating puzzle to semantic theory, particularly with two of their well-known features. First, they usually escape the effect of scope-taking operators, i.e. they **project**, as shown in (1). Second, they usually cannot be used to address the prominent Question Under Discussion (QUD), i.e. they are **not at-issue**, as shown in (2).

(1) a. Alex **didn't** invite Nate, *who is a musician*.
⇝ Nate is a musician.
b. Alex **might** invite Nate, *who is a musician*.
⇝ Nate is a musician.
c. **If** Alex invites Nate, *who is a musician*, then Mark will be happy.
⇝ Nate is a musician.

(2) a. Who had prostate cancer?
b. ??Tammy's husband, *who had prostate cancer*, was being treated at the Dominican Hospital. (AnderBois et al. 2015: ex.43)

How can a semantic characterization deliver these two features and, preferably, be implemented in a compositional fragment? Potts (2005) pioneered the now-prominent idea of a *bidimensional* semantics, according to which supplements live on a semantic dimension separate from the main, **at-issue**, content. The idea provides an account for both features in one fell swoop – the supplement-residing dimension is designated to be non-at-issue, and since it is separate from the at-issue content, no scopal interactions are possible during the semantic computation.

Such formulation of the bidimensional idea makes some strong predictions, one of which being that supplements should *always* project. It has been a frequent target of criticism, and counterexamples often contain allegedly non-projecting (thus narrow-scope) supplements. Martin

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(2017), in part following Amaral et al. (2007), pointed out that quantifiers such as *every* in the matrix clause can bind into the **anchor**, i.e. the nominal argument (immediately) precedding the NRRC on which it predicates, as in (3), or directly into the NRRC (4). Such quantificational binding, according to Martin, forces a narrow-scope reading of the supplement under the quantifier.

(3) [Every professional man I polled]^i said that while his$_i$ wife, *who had earned a bachelor's degree*, nevertheless had no work experience, he$_i$ thought she could use it to get a good job if she needed one. (Amaral et al. 2007: ex. 35)

(4) a. [Every cyclist]^i met Lance, *who gave him$_i$ a Tour de France souvenir*. (Martin 2017: ex. 13b)

b. [Every famous boxer I know]^i has a devoted brother, *who he$_i$ completely relied on back when he$_i$ was just an amateur*. (Martin 2017: ex. 16)

Schlenker (2023), following McCawley (1981); Del Gobbo (2003) and others, found arguments against bidimensionalism within cases where NRRCs are generated and *interpreted* inside the scope of conditional antecedents (5) or attitude verbs like *wonder* (6).

(5) If tomorrow I called the Chair, *who in turn called the Dean*, then we would be in big trouble.

(6) I will be wondering next Wednesday whether DSK, *who met with the judge the day before*, agreed to a settlement.

For Schlenker, sentences like (5) and (6) present both syntactic and semantic evidence for non-projecting supplements. Both of them contain future-referring past tenses, which is only possible in the syntactic scope of the subjunctive *if*, or the matrix future tense of *will*. In addition, neither (5) nor (6) entails the supplemental proposition, contrasting cases with projecting supplements as in (1).

Since bidimensionalism leads inevitably to projection (at least at this point), data such as (3) – (6) led both Martin (2017) and Schlenker (2023) to develop versions of what we will call *unidimensional* semantics. In their theories, the differences in semantic representation between at-issue and supplemental content are eliminated – the latter is now simply conjoined to the former at its attachment site, and projection and/or non-at-issueness are accounted for with additional structural (syntactic or LF) operations and pragmatic principles.

This paper defends bidimensionalism against these challenges from non-projecting supplements. We will show that not all counterarguments mentioned above are valid – in particular, Martin's examples (3) – (4) *do not* necessarily contain narrow-scope supplements. Schlenker's examples (5) – (6) do so more convincingly, but even so, the drastic turn to a unidimensional semantics may be at too high (and perhaps unnecessary) a cost of an otherwise simple explanation for both projection and non-at-issueness. Instead, we emphasize that the existence of non-projecting supplements do not necessarily mean a divorce from bidimensionalism, or, in other words, bidimensionalism is not 'cursed' by pervasive scopelessness. To show this, we develop a bidimensional semantics that retains its original perks while resolving the newly emerged problems. The new theory, largely inspired by Grove's (2019b) treatment of intensionality, encodes supplements as being composed separately from the at-issue content, but can *take scope* if needed.

The rest of the paper is organized as follows. In §2, we present and discuss the two unidimensional analyses on supplements, highlighting the difficulties they experience trying to explain projection and non-at-issueness. §3 lays out the basic architecture of the proposal, and §4 shows its application to capture different scope possibilities. Finally §5 concludes. This paper will focus on NRRCs without diving too much into other types of supplements such as nominal appositives and parentheticals (for detailed discussions, see Nouwen 2010; Sæbø 2011; Koev 2013; Schlenker 2023 and many others). However, we believe the proposal is general enough to at least have the potential of extending to the relevant cases.

2. Against unidimensionalism

2.1. Martin (2017)

Martin (2017) argued against bidimensionalism based on the assumption that quantificational binding into the NRRC (4) or its anchor (3) forces a narrow scope reading of the supplement. Since narrow-scope readings clash with the projection guaranteed by a bidimensional characterization, Martin proposed a unidimensional analysis that takes supplements to contribute incrementally to the main content via (dynamic) conjunction.[2] The analysis is illustrated below:

(7) $\underbrace{\text{Some cyclist}}_{\text{Q}}$, $\underbrace{\text{who is a doper}}_{\text{D}}$, $\underbrace{\text{won the Tour de France}}_{E}$.

(Martin 2017: ex.1, modified)

a. COMMA := λ_{QDE}. (QD) AND (THE D E)

b. (Some cyclist^i is a doper) AND (The doper$_i$ won the Tour de France).

That is, a kind of QP modification is packaged into the denotation of the comma intonation COMMA, where first the anchor Q (i.e. *some cyclist*) quantifies over the NRRC D (i.e. *who is a doper*), generating the supplement proposition that is conjoined to the matrix proposition, obtained by applying the predicate E (i.e. won the Tour de France) to the most salient discourse referent (THE D) that satisfies the supplemental property.

2.1.1. The empirical issue

The first and foremost problem with Martin (2017) is that the motivating examples (3) – (4) are not necessarily cases with narrow-scope supplements. The key observation comes from cases where a quantificational binding into the NRRC happens within a classic *scope island* such as an *if*-clause (8) or the tensed complement of *claim* (9). True quantifiers such as *every* does not scope out of these environments:

(8) If every cyclist has met Lance, Alex will given me a call.
For every cyclist *x*, if *x* met Lance, Alex will give me a call. (# $\forall >$ if)
3 (only) if every cyclist has met Lance, Alex will give me a call. (if $> \forall$)

[2] The dynamicity is not the key concern for our purpose.

(9) Someone claimed that every cyclist had met Lance.
For every cyclist x, there is some individual y (possibly different individuals for different x) that claimed that x had met Lance. (# $\forall > \exists$)
3 Some individual y claimed that for all cyclist x, x had met Lance. ($\exists > \forall$)

Now insert NRRCs. In some cases it does seem that the supplement scopes under the *if*-clause as well, as it fails to be entailed by the whole sentence:

(10) If [every cyclist]^i has met Lance, *who (then) has given him$_i$ a souvenir*, then Alex will give me a call.
$\approx$ If every cyclist has met Lance *and* received a souvenir from him, then Alex wil give me a call.
$\not\rightsquigarrow$ Lance has given every cyclist a souvenir.

However, examples where *every* binds the NRRC while the supplemental content projects out of the scope island (or at least prefers to) can be constructed. (11) shows cases with quantificational binding into the NRRC, and (12) into the anchor.

(11) a. **If** [every student]^i has bid farewell to Nate, *who had given [them]$_i$ some great advice during their individual meetings*, then we can officially close off the session.
$\rightsquigarrow$ Nate had given every student some great (and likely different) advice during their meetings.

b. Just now, the organizer **claimed** that [every boxer]^i had met Max, *who has nevertheless only been seen talking to [their]$_i$ coach about [their]$_i$ weaknesses*.
$\rightsquigarrow$ Max has only been seen talking to each boxer's coach about their (different) weaknesses (and the organizer may be making a false claim).

(12) a. **If** [every boxer]^i had brought [their]$_i$ brother, *who [they]$_i$ relied on very much as a child*, the gathering would have been heartwarming.
$\rightsquigarrow$ Every boxer relied on their brother very much as a child.

b. Just now, Cruella **claimed** that [every missing dalmatian]^i has been returned to [their]$_i$ owner, *who was just thinking that they might never see their dog again.*
$\rightsquigarrow$ Every owner of the missing dalmatians was just thinking that they might never see their dog again (and Cruela didn't claim this).

Therefore despite being universally quantified, the supplements are (or at least can be) entailed by the full clauses, and therefore have projected out of the semantic scope of the conditional antecedent (11a, 12a), and the attitude verb (11b, 12b). Quantificational binding is not sufficient to trap supplemental content inside a scope island.[3]

Furthermore, just for argument's sake, nothing really hinges on whether the scope operator creates an island – as long as the operator c-commands a universal quantifier that binds into the NRRC or the anchor, and the reading is available where a wide-scope (relative to the scope operator), supplemental universal and a narrow-scope, at-issue universal coexist, the argument would persist. This releases us from the concern of whether the scope operators we choose to

[3] Schlenker's example (5), on the other hand, shows that they are not necessary either.

make the argument are *really* scope islands (see e.g. Barker 2021). To make the point more explicit, note that *every* has been shown to occasionally scope over an embedding *make sure*, as in (13a):

(13) A student **made sure** that every invited speaker had a ride. (Farkas and Giannakidou 1996: ex.6)

a. ✓For every invited speaker *x*, a (possibly) different student made sure that *x* had a ride. ($\forall > \exists >$ make sure)

b. ✓A single student made sure that every invited speaker had a ride. ($\exists >$ make sure $> \forall$)

But even with the (at least equally possible) reading as in (13b) where *every* is interpreted under the scope of *make sure* (and therefore the indefinite subject), an NRRC that is universally quantified over by the narrow-scope *every* can still project:

(14) Someone needs to **make sure** that [every missing dalmatian]^i is returned to [their owner]$_i$, *who was just thinking that they might never see their dog again.*
$\rightsquigarrow$ For every missing dalmatian, their owner was just thinking that they might never see their dog again.
$\rightsquigarrow$ A (specific) individual needs to make sure that every dalmatian is returned (but not that what the owner was thinking).
($\forall_{\text{NRRC}} > \exists >$ made sure $> \forall$)

In short, a narrow-scope interpretation of a quantifier does not guarantee the narrow scope (or non-projection) of a NRRC within its quantificational scope. In fact, it seems that the semantic scope of a quantifier is hardly (if at all) predictive of that of its binding NRRC, therefore arguments for unidimensionalism based on quantificational configurations are likely to be inconclusive without further controls.

2.1.2. The theoretical issues

Now turn to the specifics of Martin's proposal. For him, projection is an 'epiphenomenon' of supplements *piggybacking* onto their anchors that take wide scope. Given the unidimensional analysis, the negative sentence (15) has in principle two possible LFs, (15a) and (15b). The negation scopes over the supplement in (15a), but not in (15b). But clearly (15b) is a much more preferred (if not the only) reading. To explain the contrast, Martin proposed that the anchor *Lance*, being a proper name, *prefers* to scope high, and the supplement *prefers* to be interpreted as close to its anchor as possible.

(15) **It's not the case** that Lance, *who is a doper*, won the Tour de France.

a. NOT (COMMA LANCE (A DOPER) WIN-TDF) $\rightsquigarrow \neg(\mathsf{doper}(\mathsf{l}) \wedge \mathsf{win}(\mathsf{l}))$

b. (COMMA LANCE (A DOPER))$_n$. NOT (WIN-TDF n) $\rightsquigarrow (\mathsf{doper}(\mathsf{l}) \wedge \neg\mathsf{win}(\mathsf{l}))$

This explanation faces several conceptual and empirical challenges. First, it is at least controversial whether proper names take scope at all (instead of being *scopeless*). Generally it makes no difference in interpretation whether a proper name takes wide scope or remains *in situ*, so forcing it to take wide scope at LF seems to incur a violation of scope economy (Fox 1995). Second, attributing projection solely to the scope of the anchor does not readily explain the contrast as in (16).[4] That is, if the proper name *Bill* prefers to take wide scope and there is no quantificational binding whatsoever that forces the NRRC to be interpreted *in situ*, it is hard to explain why the supplement projects in (16b) but not in (16a).

(16) a. If tomorrow I call Bill, *who in turn calls the Dean*, then we will be in big trouble.
$\not\rightsquigarrow$ The Chair will call the Dean tomorrow.
b. If tomorrow I call Bill, *who is a total jerk*, then we will be in big trouble.
$\rightsquigarrow$ The Chair is a total jerk.

Furthermore, supplement projection is clearly not restricted to only definite anchors. For example in (17), the NRRC predicates on the indefinite *a book*. Since indefinites have no problem scoping low, the LFs in (17a) and (17b) should be equally possible. Yet again only the wide-scope interpretation (17b) is possible.[5]

(17) John didn't read a book, *which Mary had recommended to him*. (AnderBois et al. 2015: ex. 72)
a. JOHN$_m$ MARY$_n$. NOT(COMMA (A BOOK) (λ_x. READ *mx*) (λ_y. RECOMMEND *ny*))
$\rightsquigarrow \neg(\exists x.(\mathsf{book}(x) \wedge \mathsf{read}(j,x) \wedge \mathsf{recommend}(m,x)))$
b. JOHN$_m$ MARY$_n$ (A BOOK)$_l$. NOT(COMMA *l* (λ_x. READ *mx*) (λ_y. RECOMMEND *ny*))
$\rightsquigarrow \exists x.\mathsf{book}(x) \wedge \neg(\mathsf{read}(j,x) \wedge \mathsf{recommend}(m,x))$

These facts suggest that at the end of the day, projection is most likely due to the innate properties of supplements, and it's not enough to just tweak the scope of their anchors. This shadows a more general problem for unidimensionalism. For bidimensionalists, the innate property that drives projection is usually just non-at-issueness, it receives a *semantic* encoding in the non-at-issue dimension and leads automatically to projection. In contrast, bidimenional approaches integrate supplements into the at-issue content, which forces them to seek extraneous reasons for projection and non-at-issueness. The task is, across the board, not trivial, as we will also see from Schlenker's attempt momentarily.

2.2. Schlenker (2023)

We take Schlenker's examples to contain genuinely narrow-scope supplements:

(5) If tomorrow I called the Chair, *who in turn called the Dean*, then we would be in big trouble.

[4]Note that (16a) is adapted from the Schlenker example (5) by taking out the 'fake' past marking, which can be taken as a grammatical factor that forces a narrow scope reading, just like quantificational binding as per Martin.
[5]Martin argued that the narrow-scope reading is blocked by the alternative containing a restrictive relative clause:

(1) John didn't read a book which Mary had recommended to him.

We acknowledge that this is a way out, but it also feeds the point that there is some inherent feature of supplements that are forcing their projection, not just the scope of their anchors.

(6) I will be wondering next Wednesday whether DSK, *who met with the judge the day before*, agreed to a settlement.

As mentioned above, the NRRCs in (5) and (6) both contain past tense that signals future times, suggesting that they must be *syntactically* embedded. Meanwhile, neither (5) nor (6) necessarily entails the supplements, suggesting that they do not project *semantically* either (see also Poschmann 2018 for experimental evidence in German). On top of that, Schlenker showed that the scope taking of NRRCs can be quite flexible, as both the *matrix*-scope reading (18), where the supplemental content projects, and the *intermediate*-scope reading (19), where it is interpreted as part of the conditional antecedent but outside the c-commanding quantifier, can be easily obtained.

(18) If tomorrow I call Bill, *who is a total jerk*, then we will be in big trouble. (same as 16b)

(19) If each of the faculty had mentioned the fact that they didn't like John, *who had gotten fired as a result*, we would now feel terrible.
(if$>$ NRRC $>$ $\forall$, Schlenker 2023: ex. 32)

Schlenker (2023) proposed a unidimensional semantics where, like in Martin (2017), supplemental content is simply conjoined to the main at-issue content. To deal with the various scope possibilities, he proposed that NRRCs can in principle be attached to any propositional node that dominates their anchor, and the conjunction happens at the attachment site. In (5), the NRRC is syntactically attached (and conjoined) to the clause *I called the Chair* inside the conditional antecedent, in (18) it is attached to the whole conditional, and in (19) it is attached to *each of the faculty had mentioned the fact that they didn't like John*.

This resort to syntax is again motivated by the morphological tense marking as in (5) – (6), which, importantly, cannot be explained at LF. But it immediately raises suspicions as (pre-Spell Out) syntactic structures, unlike LF, usually map homomorphically to surface structures; yet regardless of the projection status of the supplements, their corresponding NRRCs are always pronounced right next to their anchors in a superficially embedded position. Furthermore, if supplements can take flexible syntactic scope, one needs to explain why they (usually) don't do so in cases such as (18) and, along the same vain, why forcing a narrow-scope supplement using 'fake' past marking is not always viable, as in (20).

(20) #/?? If tomorrow I called Bill, *who was a total jerk (by the way)*, then we will be in big trouble.

Schlenker did claim that syntax does not bear the whole burden of explaining projection, and supplements carry a distinct *pragmatic* property that distinguish them from at-issue content. The property is coined **Translucency**, stated as follows:

(21) TRANSLUCENCY
  a. A supplement must make a non-trivial contribution in its local context relative to the global context C of the conversation.

b. It should be ‘easy’ to accommodate assumptions that make a supplement locally trivial, i.e. to add assumptions to C to obtain a strengthened context C+ relative to which the supplement is locally trivial.

The first condition (21a) speaks to Potts’ (2005) original observation that supplements should carry novel information, which is not relevant here. The second condition (21b) predicts, as Schlenker intended, a presupposition projection-like behavior of supplements. For instance given the non-projecting example (5), the local context of the supplement is one where the antecedent to its left, namely *tomorrow I call the Chair*, supposedly holds; as the supplement should be easily accommodated in this local context, it suggests at least a positive correlation between ‘me’ calling the Chair and the Chair calling the Dean. This prediction is more or less attested (see Schlenker 2023 and Poschmann 2018 for experimental results).

Unfortunately, even with Translucency, it is far from certain that a satisfactory explanation for the obligatory projection as in (20) would arise. There is no clear path to the general non-at-issueness either. Schlenker entertained a preliminary solution in his Appendix III. The basic idea is that, since supplements are easy to accommodate at local context due to Translucency, it should be easy for interlocutors to infer them from the current context, thus less likely for them to address QUDs. Even if this eventually leads to a full-fledged account, it has been noted that Translucency itself, especially the condition (21b) that Schlenker relied on to explain both projection and non-at-issueness, may be problematic in certain respects. Marty (2021) argued that the ‘easy-to-accommodate’ is neither necessary nor sufficient to characterize supplements. He introduced the following minimal pair:

(22) a. #Bill, *who has thirteen fingers*, speaks faster than anybody else.
b. Bill, *who has thirteen fingers*, plays arpeggios very fast.

Two things to notice. First, the supplemental content, that Bill has thirteen fingers, is quite surprising for someone who first hears about it (or at least as surprising as, say, Bill plays arpeggios very fast). Yet (22b) sounds perfectly fine. This suggests that (21b) is probably unnecessary. Second, given the same (null) context, (21a) and (21b) have the exact same supplement situated in the exact same local context, yet their acceptability differ sharply. This suggests that (21b) is insufficient, and the licensing of supplements needs to look beyond local context – possibly an evaluation of the relevance/coherence between the supplement and at-issue content. Therefore, even though an explanation for non-at-issueness might be retrieved from Translucency, Translucency itself seems to be problematic.

3. The semantics of supplements

What’s suggested by the previous section is that although a ‘radical’ bidimensional semantics like Potts’ faces difficulties accounting for non-projecting supplements, a ‘radical’ unidimensional view would have its own demon, especially when trying to explain projection and non-at-issueness. We propose to reconcile the two views by developing a bidimensional semantics where the supplemental content is generated at a separate dimension, it can be ‘discharged’ into the at-issue content *if needed*.

In recent years, an increasing body of literature has shown that **monads** (Moggi 1989; Wadler 1992) are extremely useful in integrating linguistic *side effects* (e.g. Shan 2005) into a compositional fragment that involves scope-taking (see for instance Barker 2002; Barker and Shan 2014; Giorgolo and Asudeh 2012; Charlow 2014, 2015, 2020; Grove 2019b). If one think of supplements as a kind of 'side effect', an appropriately deviced monad structure may hopefully capture their interpretive characters, while modulating its scope-taking behaviors.

A monad can be defined as a quadruple $\langle \mathsf{M}, (\cdot)^{\overset{\uparrow}{\mathsf{M}}}, (\cdot)^{\overset{\mathsf{M}}{\downarrow}}, \mu_{\mathsf{M}} \rangle$ of a type constructor M and three operations – the 'unit' operator $(\cdot)^{\overset{\uparrow}{\mathsf{M}}}$, the 'apply' operator $(\cdot)^{\overset{\mathsf{M}}{\downarrow}}$ and the 'join' operator μ_{M}. M, as a type constructor, takes a 'boring' type α and returns a 'fancy' type $\mathsf{M}\alpha$. $(\cdot)^{\overset{\uparrow}{\mathsf{M}}}$ is a polymorphic function of type $\alpha \to \mathsf{M}\alpha$ that trivially lifts values into the 'fancy' type space. $(\cdot)^{\overset{\mathsf{M}}{\downarrow}}$ is a function that enables composition in the 'fancy' type space, through which two 'fancy' types $\mathsf{M}\alpha$ and $\mathsf{M}(\alpha \to \beta)$ can be composed by applying functional application to their 'boring' cores independently of the side effects. Finally, μ_{M} (called 'join') is an operator that can collapse multiple layers of side effects into a single layer. As we will see, this operator will play an important role in determining the scope of supplements. The type signatures of the unit, apply and join operators are summarized as follows[6]:

$$
\begin{aligned}
(\cdot)^{\overset{\uparrow}{\mathsf{M}}} &:: \alpha \to \mathsf{M}\alpha \\
(\cdot)^{\overset{\mathsf{M}}{\downarrow}} &:: \mathsf{M}(\alpha \to \beta) \to \mathsf{M}\alpha \to \mathsf{M}\beta \\
\mu_{\mathsf{M}} &:: \mathsf{M}(\mathsf{M}\alpha) \to \mathsf{M}\alpha
\end{aligned}
$$

The set of monadic operations have to satisfy a series of naturality conditions (often called *'monad laws'*) that in turn ensure modular extension, i.e. monadic effects can be stacked on other (layers of) monadic effects 'for free'. We refer to Appendix A for the formulation and proof of the relevant monad laws.

We propose to model supplements using a `Reader.Set` monad, which is what Grove (2019a) used to encode intensionality. Importantly, any non-trivial supplemental content introduced by NRRCs will enter the semantic composition with (at least) two layers of monadic effects. It is due to this feature that the join operator can apply to collapse the supplemental content into the at-issue content.[7]

3.1. First layer: intensional propositions

The current analysis for supplements is built on Grove (2019b), who encoded *intensional* propositions as sets of pairs of possible worlds and truth values. For instance, the sentence *Nate left* has the following denotation:

$$\{\langle w, \mathsf{left}\,\mathbf{n}\,w\rangle \mid w \in \mathscr{W}\} \tag{a}$$

[6] The operations $(\cdot)^{\overset{\uparrow}{\mathsf{M}}}$ and $(\cdot)^{\overset{\mathsf{M}}{\downarrow}}$ comprise an *applicative functor* (McBride and Paterson 2008), a more general structure than monad.

[7] Earlier works, including Giorgolo and Asudeh (2012) and Charlow (2015), suggested using the `Writer` monad to model supplements, which basically enriches ordinary semantic values with an independently paired (supplemental) proposition. They were honest and intuitive depictions of Potts' bidimensional semantics, but also inherit the problem that the information flow is at most one-way (from the main content to the supplemental content), thus failing at capturing the various scope possibilities.

where $\mathscr{W}$ is the set of possible worlds. In other words, $\langle w, \top \rangle$ is in the set above as long as Nate left at world w, and $\langle w, \bot \rangle$ is in the set as long as Nate didn't. Conversely, the truth condition of a proposition φ, i.e. the set of worlds in which it is true, can be reconstructed from the new denotation:

(23) φ is true in w iff $\langle w, \top \rangle \in \varphi$

The compositional schema that derives such propositional meanings is powered by the monad `Reader.Set`. The representation with a *set of world-value pairs* correspond to a de-sugared form of an object of type $\mathtt{s} \to \alpha \to \mathtt{t}$, which is obtained after the `Set` component lifts the value of type α into a charateristic function of type $\alpha \to \mathtt{t}$, and the Reader component makes the new value world-sensitive. The corresponding type constructor N and *unit* are defined as follows.

$$\mathsf{N}\alpha :: \mathtt{s} \to \alpha \to \mathtt{t}$$
$$(\cdot)^{\overset{\uparrow}{\mathsf{N}}} :: \alpha \to \mathsf{N}\alpha$$
$$a^{\overset{\uparrow}{\mathsf{N}}} := \{\langle w, a \rangle \mid w \in \mathscr{W}\}$$

Then *Nate*, with an ordinary denotation $\mathbf{n}$, can be lifted by unit into $\mathbf{n}^{\overset{\uparrow}{\mathsf{N}}}$, as follows:

$$\{\langle w, \mathbf{n} \rangle \mid w \in \mathscr{W}\} \tag{b}$$

An intensional one-place property *left* has the following denotation:

$$\{\langle w, \lambda x.\, \mathsf{left}\, x\, w \rangle \mid w \in \mathscr{W} \rangle\} \tag{c}$$

Finally, note that for each pair in the denotation of *Nate left* (a), the second component can be obtained from applying simple functional application to the second components of a couple of pairs picked from (b) and (c). This is precisely what apply does when combining 'fancy'-type expressions. If we define apply $(\cdot)^{\overset{\mathsf{N}}{\downarrow}}$ as follows:

$$(\cdot)^{\overset{\mathsf{N}}{\downarrow}} :: \mathsf{N}(\alpha \to \beta) \to \mathsf{N}\alpha \to \mathsf{N}\beta$$
$$m^{\overset{\mathsf{N}}{\downarrow}} n := \{\langle w, fx \rangle \mid \langle w, f \rangle \in m \;\&\; \langle w, x \rangle \in n\}$$

Then (a) is simply the result of applying (c) to (b), as we would like it to be.

3.2. Second layer: supplements

Let's add supplements. As hinted above, we would like to make use of the join operator to model the scope-taking of supplements. The argument to which join is applied to should have a type signature of the form M(Mα), corresponding to a set of {pairs of possible worlds and sets of {pairs of possible worlds and expressions of type α}}. What supplements do, essentially, is adding to the comprehension condition of the matrix set by restricting the set of worlds that can be used as the world component in the pairs. For instance, given a phrase such as *Nate, who is a musician*, we would like it to have the following denoation:

$$\{\langle w, \{\langle w', \mathbf{n} \rangle \mid w' \in \mathscr{W}\} \rangle \mid \mathsf{musician}\, \mathbf{n}\, w\} \tag{d}$$

To derive it compositionally, I propose the following semantics for the COMMA intonation:

(24) **COMMA** :: $\mathsf{N}(\mathtt{e} \to \mathtt{t}) \to \mathtt{e} \to \mathsf{N}(\mathsf{N}\mathtt{e})$

$$:= \lambda m \lambda x. \{\langle w, \{\langle w', x\rangle \mid w' \in \mathscr{W}\}\rangle \mid \langle w, \top\rangle \in m \overset{\mathsf{N}}{\downarrow} (x^{\overset{\uparrow}{\mathsf{N}}})\}$$

That is, **COMMA** combines directly with a monadic property of type $\mathsf{N}(\mathtt{e} \to \mathtt{t})$ and an $\mathtt{e}$-type individual and returns a (doubly-lifted) $\mathsf{N}(\mathsf{N}\mathtt{e})$-type expression. Now consider the simple sentence containing an NRRC (25):

(25) Alex invited Nate, *who is a musician*.

Based on the discussion in the last section, we can expect the following semantics for *invite*:

$$\{\langle w, \lambda xy. \mathsf{invite}\, x\, y\, w\rangle \mid w \in \mathscr{W}\} \tag{e}$$

There is yet no way to compose (e) and (d), but we can use the same trick as in the last subsection. First, applying unit to lift (e) to an $\mathsf{N}(\mathsf{N}(\mathtt{e} \to \mathtt{e} \to \mathtt{t}))$-type expression:

$$(\mathrm{e})^{\overset{\uparrow}{\mathsf{N}}} = \{\langle w, \{\langle w', \lambda xy. \mathsf{invite}\, x\, y\, w'\rangle \mid w' \in \mathscr{W}\}\rangle \mid w \in \mathscr{W}\} \tag{f}$$

Then we can implement a fancier apply, which I will call apply^2 $(\cdot)\overset{\mathsf{N}^2}{\downarrow}$, to combine (f) and (d). Essentially, apply^2 uses apply $(\cdot)\overset{\mathsf{N}}{\downarrow}$ to combine the second components of selected couple of pairs from (d) and (f), while reserving the membership restrictions provided by the NRRC.[8]

$$(\cdot)\overset{\mathsf{N}^2}{\downarrow} :: \mathsf{N}(\mathsf{N}(\alpha \to \beta)) \to \mathsf{N}(\mathsf{N}\alpha) \to \mathsf{N}(\mathsf{N}\beta)$$
$$\mathbf{m} \overset{\mathsf{N}^2}{\downarrow} \mathbf{n} := \{\langle w, m \overset{\mathsf{N}}{\downarrow} n\rangle \mid \langle w, m\rangle \in \mathbf{m} \;\&\; \langle w, n\rangle \in \mathbf{n}\}$$

Then apply^2 (f) to (d), we get the denotation for *invite Nate, who is a musician*:

$$\{\langle w, \{\langle w', \lambda y. \mathsf{invite}\, \mathbf{n}\, y\, w'\rangle \mid w' \in \mathscr{W}\}\rangle \mid \mathsf{musician}\, \mathbf{n}\, w\} \tag{g}$$

Finally, feed to (h) the doubly lifted denotation for *Alex*:

$$\mathbf{a}^{\overset{\uparrow}{\mathsf{N}^2}} = \{\langle w, \{\langle w', \mathbf{a}\rangle \mid w' \in \mathscr{W}\}\rangle \mid w \in \mathscr{W}\} \tag{h}$$

We get the following denotation for (25):

$$\{\langle w, \{\langle w', \mathsf{invite}\, \mathbf{n}\, \mathbf{a}\, w'\rangle \mid w' \in \mathscr{W}\}\rangle \mid \mathsf{musician}\, \mathbf{n}\, w\} \tag{i}$$

Up to this point, the supplemental content (namely *Nate is a musician*) is processed completely independent of the main at-issue content (namely *Alex invited Nate*). The truth condition of (25) can be quite easily extracted from (i) – first, apply 'join' μ_{N}, as defined below:

$$\mu_{\mathsf{N}} :: \mathsf{N}(\mathsf{N}\alpha) \to \mathsf{N}\alpha$$
$$\mu_{\mathsf{N}}(\mathbf{m}) := \{\langle w, a\rangle \mid \exists m. \langle w, m\rangle \in \mathbf{m} \;\&\; \langle w, a\rangle \in m\}$$

[8] In fact, $(\cdot)\overset{\mathsf{N}^2}{\downarrow}$ can be derived from $(\cdot)\overset{\mathsf{N}}{\downarrow}$ based on general rules of monad transformation. See Appendix A for the proof.

That is, μ_{N} conjoins the comprehension conditions of the matrix set the set denoted by the second component of the pairs who are members in the matrix set. Applied to (i), it returns:

$$\{\langle w, \mathsf{invite}\,\mathbf{n}\,\mathbf{a}\,w\rangle \mid \mathsf{musician}\,\mathbf{n}\,w\} \tag{j}$$

Now the set contains only $\langle w, \top\rangle$ or $\langle w, \bot\rangle$, for some $w \in \mathscr{W}$. $\langle w, \top\rangle$ is in the set if and only if Nate is a musician at w (only these worlds are included), and Alex invited Nate at w (so that the second component is $\top$). Then by (23), the sentence (25) is true iff Alex invited Nate and Nate is a musician, corresponding to the logical conjunction of the at-issue and supplemental content, as desired.

To conclude this section, the derivation of (25) is presented in the following tree:

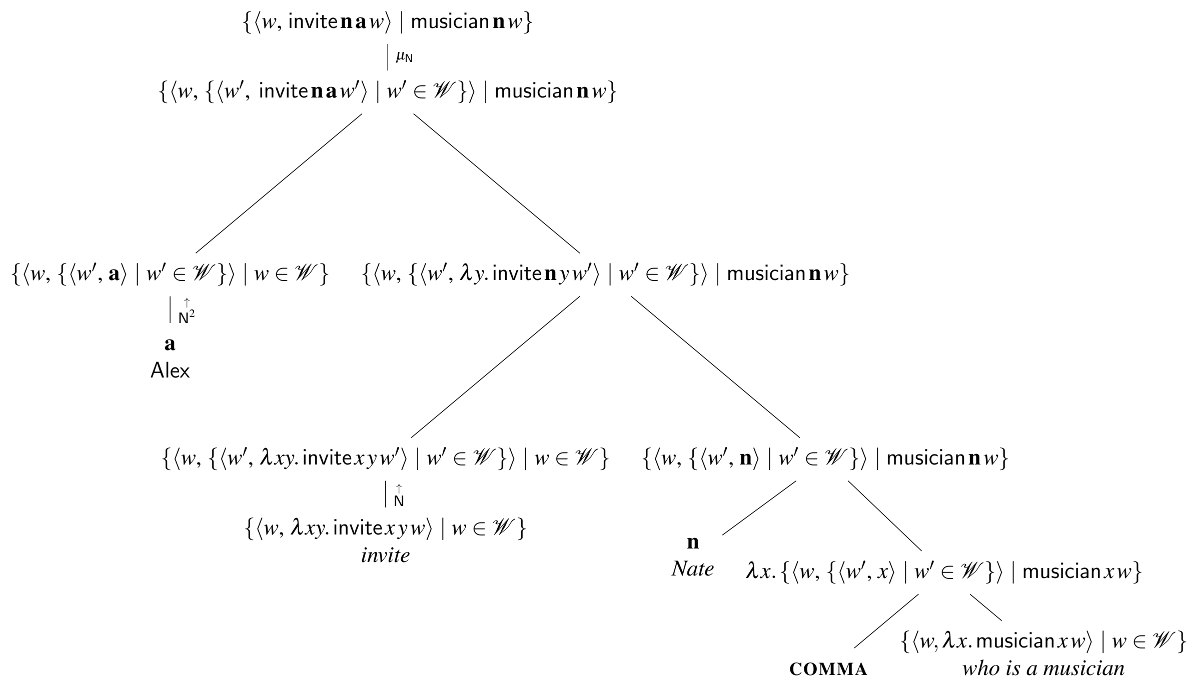

Figure 1: Alex invited Nate, who is a musician.

4. The scopes of supplements

4.1. Projection, or 'wide-scope' supplements

It is perhaps clear by now that the projection of supplement follows directly from the (default) non-interaction of at-issue and supplemental content during semantic computation. We hereby illustrate with the case where the supplement projects through the matrix-level negation negation (26), but similar derivations apply to other scope operators.

(26) Alex **didn't** invite Nate, *who is a musician*.
$\rightsquigarrow$ Nate is a musician.

Assuming the following semantics for the negation and *might*, both propositional operators:

(27) $\mathsf{not} := \lambda p.\{\langle w, \top\rangle \mid \langle w, \top\rangle \notin p\}$ $(\mathtt{Nt} \to \mathtt{Nt})$

For simplicity, it takes and returns only Nt-type arguments. In order to combine with sentences with non-trivial supplements such as *Alex invite Nate, who is a musician* denoting (i), they need to be lifted by unit $(\cdot)^{\overset{\uparrow}{\mathsf{N}}}$ before applying $(\cdot)^{\overset{\mathsf{N}}{\downarrow}}$ to (i). The derivation is shown in Figure 2.

$$\{\langle w, \top\rangle \mid \mathsf{musician}\,\mathbf{n}\,w \;\&\; \neg\mathsf{invite}\,\mathbf{n}\,\mathbf{a}\,w\}$$

$$\Big|\ \mu_{\mathsf{N}}$$

$$\{\langle w, \{\langle w', \top\rangle \mid \neg\mathsf{invite}\,\mathbf{n}\,\mathbf{a}\,w'\}\rangle \mid \mathsf{musician}\,\mathbf{n}\,w\}$$

$$\Big|\ =$$

$$\{\langle w, \lambda p.\{\langle w', \top\rangle \mid \langle w', \top\rangle \notin p\}\rangle \mid w \in \mathscr{W}\}^{\overset{\mathsf{N}}{\downarrow}}\{\langle w, \{\langle w', \mathsf{invite}\,\mathbf{n}\,\mathbf{a}\,w'\rangle \mid w' \in \mathscr{W}\}\rangle \mid \mathsf{musician}\,\mathbf{n}\,w\}$$

$$\{\langle w, \lambda p.\{\langle w', \top\rangle \mid \langle w', \top\rangle \notin p\}\rangle \mid w \in \mathscr{W}\} \qquad \{\langle w, \{\langle w', \mathsf{invite}\,\mathbf{n}\,\mathbf{a}\,w'\rangle \mid w' \in \mathscr{W}\}\rangle \mid \mathsf{musician}\,\mathbf{n}\,w\}$$

Alex invited Nate, who is a musician

$$\Big|\ \overset{\uparrow}{\mathsf{N}}$$

$$\lambda p.\{\langle w, \top\rangle \mid \langle w, \top\rangle \notin p\}$$

not

Figure 2: Alex didn't invite Nate, who is a musician.

Note that since the supplement *Nate is a musician* is logged as the comprehension condition of the matrix set, it never interacts with the negation, which is only applied to the 'inner' set, as evidenced by the final truth condition (i.e. the comprehension condition of the set on top).

4.2. Non-proejction, or 'narrow-scope' supplements

So far the join operator μ_{N} is only applied at the end of the derivation for the purpose of retrieving truth conditions. But as we suggested before, it can be applied at a sub-sentential level to 'discharge' supplements into the at-issue content. For Schlenker's example (5):

(5) If tomorrow I called the Chair, *who in turn called the Dean*, then we would be in big trouble.

μ_{N} is applied after the conditional antecedent *tomorrow I call the Chair, who in turn call the Dean* is computed, and before it is combined with *if*. Assuming the following simplified semantics for *if* (equivalent to a material conditional):

(28) $\mathsf{if} := \lambda p \lambda q.\, \mathsf{not}(p \cap \mathsf{not}\, q)$ $(\mathtt{Nt} \to \mathtt{Nt} \to \mathtt{Nt})$

The derivation of (5) proceeds as in Figure. 3. μ_{N} is applied to integrate the supplemental content, i.e. the comprehension condition of the matrix set, to that of its 'inner' set, returning an Nt-type expression which is then fed to if. Consequently, both the at-issue and (originally) supplemental components are affected by the logical operations introduced by if. In the final result, $\langle w, \top\rangle$ is a member of the resulting set iff 'I' don't call the Chair at w, or the Chair doesn't call the Dean at w, or we are in trouble at w. The truth condition is thus approximately the same as the conditional with a conjunctive antecedent as in (29), reflecting the non-projection of the supplemental content.

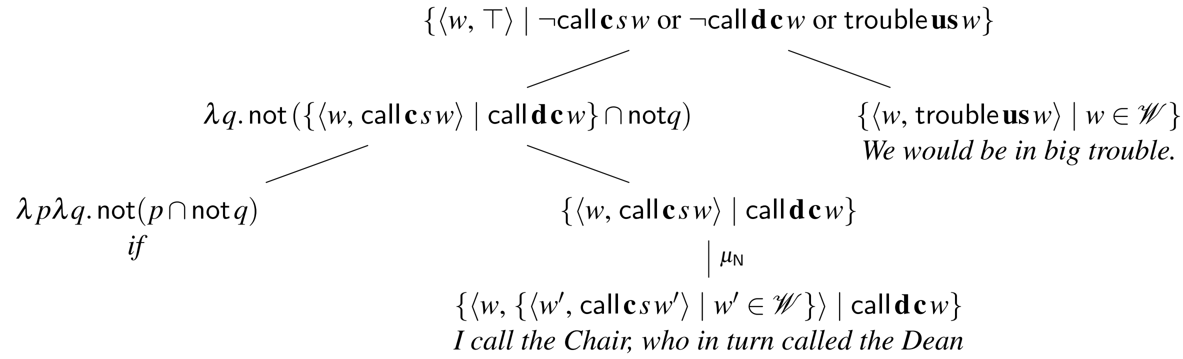

Figure 3: If tomorrow I called the Chair, who in turn called the Dean, then we would be in big trouble.

(29) If tomorrow I called the Chair and he called the Dean, then we would be in big trouble.

Finally, since the application site of μ_{N} alone determines the semantic scope of supplements, the intermediate-scope reading of supplements, as in (19), is obtained when μ_{N} is applied *after* the lower-scope quantification takes effect and *before* combining the antecedent with if. We leave it to the reader to verify the derivation.

4.3. Non-at-issueness

At this point we've accomplished what we set out do to – defending bidimensionalism against arguments based on non-projecting supplements. Supplements are registered differently than at-issue content, but can be integrated, thus take scope, when needed. The discourse effects of supplements have been mostly sidelined, to which we now turn briefly.

Recall that the hallmark of the bidimensional analysis is the coupling of non-at-issueness and projection. Since our analysis is bidimensional, with the supplemental dimension modeled as comprehension conditions of sets that asymmetrically nest sets denoted by at-issue content, the coupling is retained and the default non-at-issueness of supplements follows straightforwardly. Based on the discussions in §2, we take the simplicity of this explanation to be a good thing.

In fact, the current proposal delivers a more fine-grained account for the discourse status of supplements. Supplements do not *always* project, as they can be discharged into the computation of at-issue contents; analogously, supplements are not non-at-issue *a priori*, they may exhibit a diversity of discourse effects which are tied to their scope possibilities.

Here it is important to note that a supplement being non-projecting does not mean that the differences between its discourse effects and that of the at-issue content are eliminated. For instance, Schlenker (2023) showed that sentences with a narrow-scope supplement as in (5) carry a stronger conditional inference (in this case, that 'if tomorrow I called the Chair, he *would* call the Dean') than its minimal variation that uses explicit conjunction, as in (29). Such difference is expected due to their differences in semantic derivations, in particular that μ_{N} is applied in the derivation of (5) but not (29). The remaining question is then how exactly is the implementation of μ_{N} associated with the specific discourse effects. Leaving the detailed characterization to another occasion, it is relatively clear that narrow-scope supplements seem to be 'licensed' by some specific coherence relations. In (5) the supplement stands in a narrative/resultative, thus *coordinating* relation (Asher et al. 2003) with the adjacent at-issue content (as in 'I call the Chair and *then* he calls the Dean'), whereas for projecting supplements such as (18), the supplements tend to provide explanation or background information for the utterance (e.g. Bill being a jerk is the reason that we'll be in trouble if he knows), thus standing in a *subordinating* relation with the at-issue content. Jasinskaja and Poschmann (2021) proposed, alternatively, that narrow-scope supplements are 'licensed' when it is not speaker-oriented, while projecting ones usually are (e.g. for (18), the speaker thinks that Bill is a jerk). Regardless of what constitutes the most accurate characterization, the inferential patterns of sentences containing supplements may result from competitions between different derivations (i.e. those involve sub-sentential application of μ_{N} that those do not) in terms of maximizing discourse coherence.

5. Conclusion

In this paper, we defended the traditional bidimensional approach to supplements *à la* Potts (2005) by developing a version that allows supplements to take 'narrow scope' and enter the computation of at-issue content, thus resolving the problems pertaining to the existence of non-projecting supplements. The new semantics lays the foundation for a more fine-grained characterization of the discourse effects of supplements.

Note that the theoretical objective of the paper is simply reconciling bidimensionalism with one class of criticisms, i.e. non-projecting supplements. Other types of challenges remain. For instance, the semantic interactions between at-issue and supplemental components extend well beyond scoping towards empirical domains such as anaphora resolution, presupposition satisfaction and ellipsis (see e.g. Nouwen 2007; Amaral et al. 2007; AnderBois et al. 2015). Others challenge bidimensionalism by questioning the virtue of non-at-issueness directly (e.g. Koev 2013, 2015). It is beyond the scope of this paper to address these issues, but we believe that the current proposal is amenable enough to reach the ultimate goal. We've already shown in the previous section that bidimensionalism does not necessitates non-at-issueness (or projection) of supplements, which gives us the space of accommodating more empirical variations. For the semantic interactions, the common strategy (as used in AnderBois et al. 2015) is to deploy a dynamic framework, and it turns out that a monadic structure can be extended to a dynamic representation quite effortlessly (Charlow 2014). In general, therefore, we hope that this paper has provided an improved semantic basis for a more accurate depiction of supplements.

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A. Proof of Monad Laws

Monad laws can be formulated in several equivalent ways. In order to give a more transparent representation, we first define a new **sequencing operator** $\multimap$ from the general quadruple definition $\langle \mathsf{M}, (\cdot)^{\overset{\uparrow}{\mathsf{M}}}, (\cdot)^{\overset{\mathsf{M}}{\downarrow}}, \mu_{\mathsf{M}} \rangle$.

$$\begin{aligned} \multimap_{\mathsf{M}} &:: \mathsf{M}\alpha \to (\alpha \to \mathsf{M}\beta) \to \mathsf{M}\beta \\ &:= \lambda \mathbf{a}.\lambda \mathbf{f}.\, \mu_{\mathsf{M}}((\mathbf{f}^{\overset{\uparrow}{\mathsf{N}}})^{\overset{\mathsf{N}}{\downarrow}} \mathbf{a}) \end{aligned}$$

Monad laws. Monad laws as formulated in Charlow (2014):

$$\begin{aligned} a^{\overset{\uparrow}{\mathsf{N}}} \multimap k &= ka && \text{LeftID} \\ \mathbf{m} \multimap (\cdot)^{\overset{\uparrow}{\mathsf{N}}} &= \mathbf{m} && \text{RightID} \\ (\mathbf{m} \multimap k) \multimap c &= \mathbf{m} \multimap (\lambda a.ka \multimap c) && \text{Associativity} \end{aligned}$$

Now given the specific $\mathsf{Reader.Set}$ monad N, a type-$\mathsf{N}\alpha$ expression **a** should be in the form $\{\langle w, a_\alpha \rangle \mid \textsc{Cond}_A\}$, and a type $\alpha \to \mathsf{M}\beta$ expression **f** in the form $\lambda x_\alpha.\{\langle w, (fx)_\beta \rangle \mid \textsc{Cond}_B\}$. The sequencing operator $\multimap_{\mathsf{N}}$ is thus:

$$\begin{aligned} \multimap_{\mathsf{N}} :&= \lambda \mathbf{a}_{\mathsf{N}\alpha}.\lambda \mathbf{f}_{\alpha \to \mathsf{N}\beta}.\, \mu_{\mathsf{N}}((\mathbf{f}^{\overset{\uparrow}{\mathsf{N}}})^{\overset{\mathsf{N}}{\downarrow}} \mathbf{a}) \\ &= \lambda \mathbf{a}.\lambda \mathbf{f}.\, \mu_{\mathsf{N}}(\{\langle w, \lambda x_\alpha.\{\langle w', (fx)_\beta \rangle \mid \textsc{Cond}_B[w']\}\rangle \mid w \in \mathscr{W}\}^{\overset{\mathsf{N}}{\downarrow}} \{\langle w, a_\alpha \rangle \mid \textsc{Cond}_A[w]\}) \\ &= \lambda \mathbf{a}.\lambda \mathbf{f}.\, \mu_{\mathsf{N}}(\{\langle w, \{\langle w', fa \rangle \mid \textsc{Cond}_B[w']\}\rangle\ \textsc{Cond}_A[w]\}) \\ &= \lambda \mathbf{a}.\lambda \mathbf{f}.\, \{\langle w, fa \rangle \mid \textsc{Cond}_A[w]\ \&\ \textsc{Cond}_B[w]\} \end{aligned}$$

The unspecified $\textsc{Cond}_A$ and $\textsc{Cond}_B$ will cause representational inconvenience later. An equivalent presentation is as follows (read as 'm evaluates the variable v in program π', where m, v and π are of the type $\mathsf{N}\alpha$, α, $\mathsf{N}\beta$):

$$m_v \multimap \pi = \{\langle w, \pi_1[m_1/v]\rangle \mid \langle w, m_1 \rangle \in m\ \&\ \langle w, \pi_1 \rangle \in \pi\}$$

$\pi_1[m_1/v]$ means substituting all the occurrences of v in π_1 with m_1. Associativity can be rewritten accordingly:

$$(m_v \multimap k)_u \multimap c = m_v \multimap (k_u \multimap c)$$

Proof.

$$a^{\overset{\uparrow}{\mathsf{N}}} \multimap k = \{\langle w, a\rangle \mid w \in \mathscr{W}\} \multimap k$$
$$= \bigcup_{\langle w,a\rangle \in \{\langle w,a\rangle \mid w \in \mathscr{W}\}} ka \qquad \text{(Left ID)}$$
$$= ka \qquad (a \text{ does not covary with } w)$$

$$\mathbf{m} \multimap (\cdot)^{\overset{\uparrow}{\mathsf{N}}} = \bigcup_{\langle w,m\rangle \in \mathbf{m}} m^{\overset{\uparrow}{\mathsf{N}}}$$
$$= \bigcup_{\langle w,m\rangle \in \mathbf{m}} \{\langle w, m\rangle \mid w \in \mathscr{W}\} \qquad \text{(Right ID)}$$
$$= \mathbf{m} \qquad (w \in \mathscr{W} \text{ is a trivial membership condition})$$

$$(m_v \multimap k)_u \multimap c = \{\langle w, k_1[m_1/v]\rangle \mid \langle w, k_1\rangle \in k \;\&\; \langle w, m_1\rangle \in m\}_u \multimap c$$
$$= \{\langle w, c_1[k_1[m_1/v]/u]\rangle \mid \langle w, k_1\rangle \in k \;\&\; \langle w, m_1\rangle \in m \;\&\; \langle w, c_1\rangle \in c\}$$

$$m_v \multimap (k_u \multimap c) = m_v \multimap \{\langle w, c_1[k_1/u]\rangle \mid \langle w, k_1\rangle \in k \;\; \langle w, c_1\rangle \in c\} \qquad \text{(Associativity)}$$
$$= \{\langle w, c_1[k_1/u][m_1/v]\rangle \mid \langle w, k_1\rangle \in k \;\&\; \langle w, m_1\rangle \in m \;\&\; \langle w, c_1\rangle \in c\}$$
$$= \{\langle w, c_1[k_1[m_1/v]/u]\rangle \mid \langle w, k_1\rangle \in k \;\&\; \langle w, m_1\rangle \in m \;\&\; \langle w, c_1\rangle \in c\}$$
$$= (m_v \multimap k)_u \multimap c \; (\text{only } k_1 \text{ contains the variable } v \text{ after } \alpha\text{-conversion})$$

Monad Transformation. Now we show that the definition of $(\cdot)^{\overset{\mathsf{N}^2}{\downarrow}}$ can be derived by applying a *monad transformers* based on N back on N. Given an arbitrary monad $\langle \mathsf{M}_1, (\cdot)^{\overset{\uparrow}{\mathsf{M}_1}}, \multimap_1\rangle$, applying the monad transformation based on N gives rise to another monad $\langle \mathsf{M}_2, (\cdot)^{\overset{\uparrow}{\mathsf{M}_2}}, \multimap_2\rangle$:

$$\mathsf{M}_2\alpha ::= s \to \mathsf{M}_1\alpha \to t$$
$$a^{\overset{\uparrow}{\mathsf{M}_2}} := \{\langle w, a^{\overset{\uparrow}{\mathsf{M}_1}}\rangle \mid w \in \mathscr{W}\}$$
$$m_v \multimap_2 \pi := \{\langle w, m'_u \multimap_1 \pi_1[u/v]\rangle \mid \langle w, \pi_1\rangle \in \pi \;\&\; \langle w, m'\rangle \in m\}$$

Now suppose $\mathsf{M}_1 = \mathsf{N}$, we get the following M_2:

$$\mathsf{M}_2\alpha ::= s \to (s \to \alpha \to t) \to t$$
$$a^{\overset{\uparrow}{\mathsf{M}_2}} := \{\langle w, \{\langle w', a\rangle \mid w' \in \mathscr{W}\}\rangle \mid w \in \mathscr{W}\}$$
$$m_v \multimap_2 \pi := \{\langle w, \{\langle w', \pi_2[m''/u]\rangle \mid \langle w', \pi_2\rangle \in \pi_1[u/v] \;\&\; \langle w', m''\rangle \in m'\}\rangle \mid \langle w, \pi_1\rangle \in \pi \;\&\; \langle w, m'\rangle \in m\}$$
$$= \{\langle w, \{\langle w', \pi_2[m''/v]\rangle \mid \langle w', \pi_2\rangle \in \pi_1 \;\&\; \langle w', m''\rangle \in m'\}\rangle \mid \langle w, \pi_1\rangle \in \pi \;\&\; \langle w, m'\rangle \in m\}$$

We can show that $(\cdot)^{\overset{\mathsf{N}^2}{\downarrow}}$ can be defined with $\multimap_2$ and $(\cdot)^{\overset{\uparrow}{\mathsf{M}_2}}$:

$$(\cdot)^{\overset{\mathsf{N}^2}{\downarrow}} = \lambda u_{\mathsf{M}_2\alpha} \lambda v_{\mathsf{M}_2(\alpha\to\beta)}.\, v_{f_{\alpha\to\beta}} \multimap_2 (u_{x_\alpha} \multimap_2 (fx)^{\overset{\uparrow}{\mathsf{M}_2}})$$

Proof.

$$u \overset{\mathsf{N}^2}{\downarrow} v = \{\langle w', v_1 \overset{\mathsf{N}}{\downarrow} u_1\rangle \mid \langle w', u_1\rangle \in u \;\&\; \langle w', v_1\rangle \in v\}$$

$$= \{\langle w', \{\langle w, v_2\ u_2\rangle \mid \langle w, u_2\rangle \in u_1 \;\&\; \langle w, v_2\rangle \in v_1\}\rangle \mid \langle w', u_1\rangle \in u \;\&\; \langle w', v_1\rangle \in v\}$$

$$v_f \multimap_2 (u_x \multimap_2 (fx)^{\overset{\uparrow}{\mathsf{M}_2}}) = v_f \multimap_2 \{\langle w, \{\langle w', fu_2\rangle \mid \langle w', u_2\rangle \in u_1\}\rangle \mid \langle w, u_1\rangle \in u\}$$

$$= \{\langle w, \{\langle w', v_2 u_2\rangle \mid \langle w', u_2\rangle \in u_1 \;\&\; \langle w', v_2\rangle \in v_1\}\rangle \mid \langle w, u_1\rangle \in u \;\&\; \langle w, v_1\rangle \in v\}$$

$$= u \overset{\mathsf{N}^2}{\downarrow} v$$