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Variance estimation for sensitivity analysis of poverty and inequality measures

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Estimates of poverty and inequality are often based on application of a single equivalence scale, despite the fact that a large number of different equivalence scales can be found in the literature. This paper describes a framework for sensitivity analysis which can be used to account for the variability of equivalence scales and allows to derive variance estimates of results of sensitivity analysis. Simulations show that this method yields reliable estimates. An empirical application reveals that accounting for both variability of equivalence scales and sampling variance leads to confidence intervals which are wide.

Keywords: equivalence scale; influence function; low income proportion; sensitivity analysis; variance estimation

1 Introduction

Equivalence scales play a major part in research on poverty and inequality. They are used to adjust household income of households of different size and composition for differences in relative costs of reaching the same living standard. The result is called equivalent income. Equivalent income can be directly compared across households and is used to calculate measures of poverty and inequality. A well-known equivalence scale is the modified OECD scale (Hagenaars, de Vos, & Zaidi, 1994), which finds widespread use in economics, sociology, statistics, and other disciplines. Apart from the modified OECD scale other popular scales exist, like the equivalence scale suggested by McClements (1977), which is a common choice for analysis of British data. In general, a large variety of equivalence scales can be found in the literature, due to different estimation methods and different assumptions. This raises the question which of the many available scales one should use, because results are sensitive to the choice of equivalence scale (Buhmann, Rainwater, Schmaus, & Smeeding, 1988; Burkhauser, Smeeding, & Merz, 1996; Coulter, Cowell, & Jenkins, 1992; De Vos & Zaidi, 1997; Székely, Lustig, Cumpa, & Mejía, 2004). Estimating an equivalence scale for each specific research question and data set would avoid this choice, but is infeasible, as estimation of equivalence scales is a time consuming task and there is no consensus on which of the many available methods are preferable (Muellbauer & van de Ven, 2004).

Using a single equivalence scale taken from the literature, as is commonly done, ignores that there is a wide range of equivalence scales of which none can be claimed to be superior to others, as each method for derivation of equivalence scales has its advantages and disadvantages (see e.g. Coulter et al., 1992; Schröder, 2013; Schulte, 2007). Applying a single scale leads to valid estimates of poverty and inequality conditional on the chosen equivalence scale, but the choice of the equivalence scale may be hard to justify. Sensitivity analyses are a simple remedy, which means calculating the indicators of interest using alternative equivalence scales (Gustafsson, 1995). In many empirical applications no sensitivity analysis is conducted, though. Moreover, the sample variance of the results of sensitivity analysis has been ignored in the literature and variance estimates are only calculated for the results of the main analysis, if at all.

The main contribution of this paper is to propose a framework for sensitivity analysis which allows to include variability of equivalence scales in a formal way and which yields a simple method for variance estimation for results of sensitivity analysis. Using a certain parametrization of equivalence scales the researcher specifies a univariate probability distribution which describes the set of possible equivalence scales. In combination with sample data this induces a distribution for a measure of poverty or inequality. The induced distribution can be analysed using its mean, median, extrema, and so forth. Linearization techniques based on influence functions as proposed by Deville, 1999 are applied to derive variance estimates. The simultaneous confidence interval of the minimum and the maximum of the induced distribution is proposed as a way of capturing both variability of equivalence

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scales and sample variance. In doing so, this paper brings together the literature on sensitivity analysis and the literature on variance estimation. The former has ignored sample variance so far and the latter ignored variability of equivalence scales.

There is a large number of measures of poverty and inequality (see e.g. Zheng, 1997). As an important example, the low income proportion will be analyzed, which is usually defined as the proportion of households with equivalent income equal or less than 60% of median equivalent income. This indicator, also known as at-risk-of-poverty rate, is routinely reported by national statistical offices. It is part of the EU2020 indicators which are used to measure the achievement of policy goals in the European Union and its member states. As such, the low income proportion has received considerable attention in the methodological literature on variance estimation for non-linear indicators and surfaces regularly in public debates. The results derived in this paper apply to other measures as well, as long as certain assumptions hold, and additional results for the Gini index and the quintile share ratio can be found in the supplementary materials.

The remainder of this paper is structured as follows. Section 2 introduces basic concepts and notation. An easy way to formalize assumptions of sensitivity analysis is explained in section 3. A discussion of linearization for variance estimation with equivalent income assumed known (i.e. using one specific equivalence scale) is given in section 4. In section 5 the framework for variance estimation for sensitivity analysis is discussed. Simulation results relating to the performance of the variance estimators are presented in section 6. Results of an empirical application to German data are provided in section 7. Section 8 concludes.

2 Notation

2.1 Low income proportion

Let $\mathcal{U} = \{1,\dots,k,\dots N\}$ be a finite population of households of size N. Y^* denotes equivalent income and y_k^* denotes equivalent income of household k. The cumulative distribution function (cdf) of Y^* is given by

$$F_{Y^*}(z) = \frac{1}{N}\sum_{k\in\mathcal{U}} \mathbb{1}(y_k^* \leq z), \tag{1}$$

with $\mathbb{1}(\cdot)$ being the indicator function. The βth quantile of Y^* is given by

$$Q_{Y^*}(\beta) = F_{Y^*}^{-1}(\beta) = \inf\{z : F_{Y^*}(z) \geq \beta\}. \tag{2}$$

For example, $Q_{Y^*}(0.5)$ is the median.

The low income threshold is defined as $\alpha Q_{Y^*}(\beta)$. β is usually set to 0.5 and common choices for α are 0.6 or 0.5. Given a choice of values for α and β the low income proportion is defined as

$$\begin{aligned} P(\alpha,\beta) &= F_{Y^*}(\alpha Q_{Y^*}(\beta)) \\ &= \frac{1}{N}\sum_{k\in\mathcal{U}} \mathbb{1}(y_k^* \leq \alpha Q_{Y^*}(\beta)). \end{aligned} \tag{3}$$

Given a sample $\mathcal{S} \subset \mathcal{U}$ of size n, $F_{Y^*}(z)$ is estimated via its sample analogue $\hat{F}_{Y^*}(z) = 1/n\sum_{k\in\mathcal{S}} \mathbb{1}(y_k^* \leq z)$ in case of simple random sampling. Otherwise $\hat{F}_{Y^*}(z) = 1/\hat{N}_w \sum_{k\in\mathcal{S}} w_k \mathbb{1}(y_k^* \leq z)$ is used, with survey weights w_k and $\hat{N}_w = \sum_{k\in\mathcal{S}} w_k$. The estimates $\hat{Q}_{Y^*}(\beta)$ and $\hat{P}(\alpha,\beta)$ of $Q_{Y^*}(\beta)$ and $P(\alpha,\beta)$ can be found by replacing F_{Y^*} in equations (2) and (3) with $\hat{F}_{Y^*}$.[1]

2.2 Equivalence scales

Equivalent income Y^* is not directly observed, but derived from observed income Y. Let $h = 1,\dots,H$ index different household types, each with a specific equivalence weight A_h. A_h is set to 1 for one household type chosen as reference, e.g. single person households. Equivalent income of all other households is standardized to be comparable to the reference household type. In what follows it will be assumed that households are only differentiated by household size and that h will denote the number of household members. An equivalence scale is defined as a set of equivalence weights for all household types, $\mathcal{A} = (A_1,\dots,A_H)$. Equivalent income for household k of size h is calculated via $y_k^* = y_k/A_h$. $\mathcal{U}_h$ and $\mathcal{S}_h$ are subsets of $\mathcal{U}$ and $\mathcal{S}$ each including all households of size h.

Replacing equation (1) with

$$F_{Y^*}(z) = \frac{1}{N}\sum_{h=1}^{H}\sum_{k\in\mathcal{U}_h} \mathbb{1}(y_k/A_h \leq z) \tag{4}$$

and plugging this definition into equations (2) and (3) the low income proportion is given by

$$P(\alpha,\beta) = \frac{1}{N}\sum_{h=1}^{H}\sum_{k\in\mathcal{U}_h} \mathbb{1}\left(y_k/A_h \leq \alpha \inf\left\{z : \left[\frac{1}{N}\sum_{h=1}^{H}\sum_{k\in\mathcal{U}_h} \mathbb{1}(y_k/A_h \leq z)\right] \geq \beta\right\}\right). \tag{5}$$

Changing equivalence weights can affect the low income proportion through both the low income threshold and the number of households below the threshold.

[1] The low income proportion can also be defined with respect to individuals. For example, one could be interested in the proportion of children living below the low income threshold. The low income proportion is then given by $\frac{1}{N_c}\sum_{k\in U} c_k \mathbb{1}(y_k^* \leq \alpha Q(\beta))$, where c_k is the number of children in household k and N_c is the total number of children. Variance estimation can proceed as described in sec. 3 and sec. 4, but has to take into account additional sampling variability through c_k (see e.g. Thuysbaert, 2008).

3 Formalizing assumptions of sensitivity analysis

In sensitivity analyses usually a limited number of additional equivalence scales $\mathcal{A}_1, \mathcal{A}_2, \ldots$ is used and analyses are rerun (e.g. Streak, Yu, & Van der Berg, 2009; Székely et al., 2004; Triest, 1998).[2] A more formal approach can start from distributional assumptions about the elements of $\mathcal{A}$ summarized in some density $f_{\mathcal{A}}$. The choice of $f_{\mathcal{A}}$ is subjective and depends on what the researcher is willing to justify. As a simple way to specify $f_{\mathcal{A}}$ the representation of equivalence scales given by Buhmann et al., 1988 will be used, which only needs one parameter to be specified to yield a complete equivalence scale, making it a useful tool for sensitivity analysis. Let h denote the number of household members. Equivalence weights are calculated via

$$A_h = h^{\eta} \tag{6}$$

with $0 \leq \eta \leq 1$. Setting $\eta = 1$ leaves h unchanged and gives the head-count-ratio, whereas $\eta = 0$ amounts to leaving household income unmodified. This representation is generally considered to approximate most equivalence scales quite well, even equivalence scales which are based on several parameters, e.g. different weights for adults and children. For instance, the modified OECD scale can be approximated by setting $\eta = 0.54$. Figure 1a shows how equivalence weights A_h for households of size $h = 1, \ldots, 5$ depend on η (dashed and dotted lines). Increasing η increases equivalence weights for all household types. Equivalence weights of the modified OECD scale are shown as points at $\eta = 0.54$ and are close to the lines, confirming the good fit of the approximation via equation (6).[3]

Using equation (6) only distributional assumptions for η are needed and an univariate distribution suffices to specify $f_{\mathcal{A}}$. This distribution can either be derived from the literature or set ad hoc based on plausibility. For example, based on an overview of equivalence scale estimates for Germany given by Schulte, 2007, which covers different methods and data sets, Dudel, Garbuszus, Ott, and Werding, 2013 arrive at values of η which are shown in figure 1b. Each dot represents a value of η obtained for a specific equivalence scale.

Values of η range from 0.32 to 0.72. These values suffice to specify a uniform distribution, i.e. $\text{Unif}(0.32, 0.72)$, based on the assumption that the resulting range covers all plausible values and none of the values in the specified interval is more probable than another. As the three uppermost values of η shown in figure 1b could be interpreted as outliers, one could restrict attention to values of η based on equivalence scales which were estimated using expenditure data. These range from 0.34 to 0.51 and one could assume that $\eta \sim \text{Unif}(0.34, 0.51)$, which seems more reasonable than $\text{Unif}(0.32, 0.72)$. Given the findings in figure 1b other distributions may be hard to justify. Still, other distributions with limited support could be used instead and the supplementary materials include simulation results based on a truncated normal distribution. Using continuous distributions seems more appropriate than discrete distributions, as the values of η shown in figure 1b are not “exact” due to sampling variability, because all of the underlying equivalence scales taken from the literature are based on sample data.

For other countries than Germany assumptions on η can be derived from the literature in a similar fashion. While one should be careful with using the numbers given above for other countries, this would nevertheless be possible if economies of scale and scope can be assumed to be similar to those in Germany, i.e. if the relative costs of adding a new member to a household are similar.

4 Variance estimation via linearization

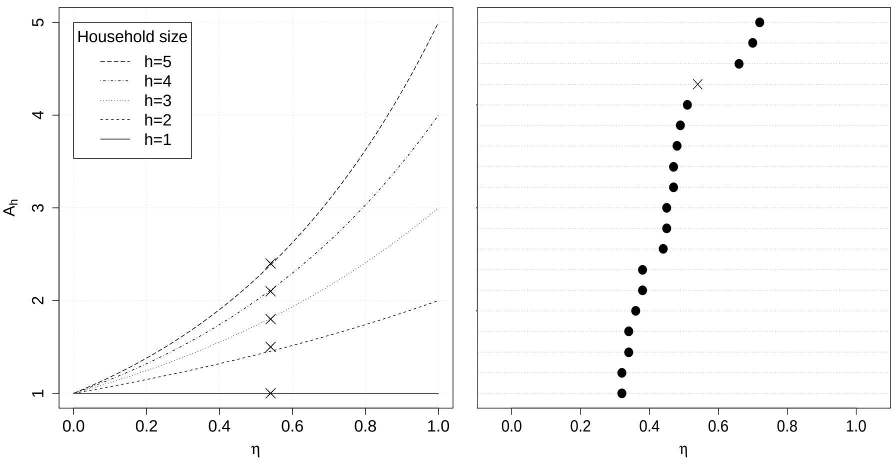
Variance estimation for the low income proportion can not use standard approaches for proportions, because the low income threshold in equation (3) is endogenous and has to be estimated from the data. More generally, most measures of poverty and inequality are nonlinear functions of the data and variance estimation is not straightforward. Variance estimation for nonlinear measures has received considerable attention in the literature and one approach is to use linearization. Variance estimation for the low income proportion with some equivalence scale assumed given has been discussed by Shao and Rao, 1993, Binder and Kovacevic, 1995, Preston, 1995, Deville, 1999, Zheng, 2001, Berger and Skinner, 2003, Osier, 2009, and Graf and Tillé, 2014, using different linearization techniques and different assumptions on sample design. Their main results concerning variance estimation for the low income proportion coincide, though, and in what follows the approach of Deville, 1999 will be described.

Suppose $\hat{P}(\alpha, \beta)$ has been estimated via sample data. Interest lies in calculating the variance $\text{Var}(\hat{P})$. This can be achieved by calculating a linearized variable z_k and estimating the variance of $t_z = \sum w_k z_k$. $\text{Var}(t_z)$ can be calculated by standard methods, e.g. the Horvitz-Thompson estimator. Asymptotic arguments establish that $\text{Var}(\hat{P}) \approx \text{Var}(t_z)$ (for details see Deville, 1999).

The values of z_k can be found through the influence function. Let X be a variable of interest and let M be a discrete measure with unit mass for each point x_k and a total mass of

[2] Notable exceptions which explore the dependence of results on equivalence scales more in-depth include Duclos and Mercader-Prats, 1999, Burkhauser et al., 1996, Banks and Johnson, 1994 and Coulter et al., 1992.

[3] For equivalence weights of the modified OECD scale shown in figure 1a it was assumed that the second person in a household is above age 14, while the third, fourth, and fifth person are below age 14. This leads to equivalence weights of 1 (h=1), 1.5 (h=2), 1.8 (h=3), 2.1 (h=4), and 2.4 (h=5). η was derived by using these equivalence weights as well as the weight for a single parent with one child (1.3) as the dependent variable in the non-linear regression $A_h = h^{\eta} + e_h$.



(a) (b)

Figure 1. (a) Equivalence weights for households of size $h = 1, \dots, 5$ as functions of η (lines) and equivalence weights of the modified OECD scale (crosses at $\eta = 0.54$); (b) Values of η obtained from the literature reviews of Schulte, 2007 and Dudel, Garbuszus, Ott, and Werding, 2013 (points) and modified OECD scale (cross at $\eta = 0.54$);

N. Let $T(M)$ be a functional of M. For example, the cdf of X can be written as $F_X(z) = [1/\int dM]\int \mathbb{1}(x \leq z)dM$. The influence function of a functional $T(M)$ as defined by Deville is given by

$$I_T(M;x) = \lim_{\epsilon \to 0} \frac{T(M+\epsilon\delta_x) - T(M)}{\epsilon}, \tag{7}$$

where δ_x is the Dirac measure at x. I_T is the Gateaux differential of T in direction of δ_x. It captures the change in T given an infinitesimal change in M. Inserting x_k into $I_T(M;\cdot)$ gives z_k. Thus, as Graf and Tillé, 2014 note, Deville's approach starts from the population parameter $T(M)$ and not the sample estimator $T(\hat{M})$. However, $T(M)$ is not known and is estimated by its sample analogue $T(\hat{M})$. $T(\hat{M})$ is plugged into equation (7), giving an estimate of z_k, $\hat{z}_k$, which can be used to approximate the variance of $T(\hat{M})$ via $\mathrm{Var}[T(\hat{M})] \approx \mathrm{Var}(\sum_{k\in S} w_k\hat{z}_k)$ (see Deville, 1999 for details).

Deville, 1999 shows that influence functions follow the standard rules of differential calculus and gives a set of useful rules which can be used to derive influence functions. Osier, 2009 provides detailed derivations for many indicators including the low income proportion. The influence function of $P(\alpha,\beta)$ is given by

$$\begin{aligned} I_P(M;y^*) = {} & \frac{1}{N}\left[\mathbb{1}(y^* \leq \alpha Q_{Y^*}(\beta)) - P(\alpha,\beta)\right] - \\ & \alpha\frac{1}{N}\frac{f_{Y^*}(\alpha Q_{Y^*}(\beta))}{f_{Y^*}(Q_{Y^*}(\beta))}\left[\mathbb{1}(y^* \leq Q_{Y^*}(\beta)) - \beta\right], \end{aligned} \tag{8}$$

where $f_{Y^*}(z)$ is the derivative of $F_{Y^*}(z)$, i.e. the probability density function (pdf) of Y^*. In Deville, 1999 multiplication of the second term by α is missing. Variants of this result can also be found in Shao and Rao (1993, p. 401), Binder and Kovacevic (1995, p. 142), Preston (1995, p. 95), Zheng (2001, p. 345) and Osier (2009, p. 173). As noted by Preston, 1995 the first term covers the variance of P for a fixed low income threshold and the second term captures the effect of estimating the threshold from the data. An analysis carried out by Zheng, 2001 indicates that variance estimates acknowledging estimation of the threshold generally tend to be higher than estimates only based on the first term, because the variance of the low income proportion with an estimated poverty threshold is higher. For estimation of the linearized variable the sample analogs $\hat{Q}_{Y^*}(\beta)$, $\hat{P}(\alpha,\beta)$ and $\hat{f}_{Y^*}(\cdot)$ are plugged into equation (8).

Because $F_{Y^*}(z)$ as defined by equation (1) is a step function, the pdf $f_{Y^*}(z)$ equals 0 or does not exist. One solution is to replace summation with integration and assume that $F_{Y^*}(z)$ is continuously differentiable (e.g. Zheng, 2001). Deville, 1999 proposed to use a convoluted smooth function instead of the non-smooth function, i.e. a smoothed version of $F_{Y^*}(z)$ to achieve continuous differentiability. A theoretical justification was given by Wang and Opsomer, 2011 using kernel estimators. Their approach requires certain smoothness and tail properties of the kernel estimator, which are satisfied by popular kernel functions like triangle and Gaussian kernels. Usually the latter is used, either for estimation of $\tilde{F}_{Y^*}(z)$ or for direct estimation of $f_{Y^*}(z)$ if $F_{Y^*}(z)$ is assumed to be differentiable (e.g. Berger & Skinner, 2003; Münnich & Zins, 2011; Osier, 2009; Preston, 1995). The simulations and the empirical application which will be presented in this paper are based on the assumption that $F_{Y^*}(z)$ is continuously differentiable. Finite sample performance of this approach is quite good at least for large samples, while the use of the convolution product as proposed by Deville, 1999 changes the quantity to be estimated and is not easy to interpret.

5 Variance estimation for induced distributions of P

5.1 Induced distributions of P

Let $f_\eta(u)$ be the pdf of the distribution chosen for η. In combination with $\mathcal{U}$ this induces a distribution of the low income proportion, $f_P(p)$. This means that for every equivalence scale $\mathcal{A} = h^\eta$ on the support of f_η $P(\alpha,\beta|\eta)$ can be calculated, resulting in a distribution of $P(\alpha,\beta)$ which depends on the choice of f_η. Switching to the case of a sample $\mathcal{S}$, a natural estimator of $f_P(p)$ is $f_{\hat{P}}(p)$.[4] An example of such a distribution $f_{\hat{P}}$ based on the data described in the next section and using Unif(0, 1) for η is shown in figure 2b. Figure 2a shows the low income proportion as a function of η, highlighting the strong impact of η on the low income proportion.[5] The choice of Unif(0, 1) may be rather extreme and more realistic examples will be used in sections 6 and 7.

The induced distribution of $P(\alpha,\beta)$ can be characterized through its expectation, quantiles, and the likes. For example, the expectation E(P) can be written as

$$\mathrm{E}(P) = \int P(\alpha,\beta|u) f_\eta(u) du, \tag{9}$$

where $P(\alpha,\beta|u)$ is given by replacing $F_{Y^*}(z)$ in equations (2) and (3) with

$$F_{Y^*}(z;u) = \frac{1}{N} \sum_{h=1}^{H} \sum_{k \in \mathcal{U}_h} \mathbb{1}(y_k/h^u \leq z). \tag{10}$$

Estimation can simply proceed by replacing $F_{Y^*}(z;u)$ with its sample analogue $\hat{F}_{Y^*}(z;u)$, calculating $P(\alpha,\beta|u)$ on the support of f_η, and weighting by f_η. Note that E(P) does not necessarily equal $P(\alpha,\beta|\mathrm{E}(\eta))$ even if the distribution of η is assumed to be symmetric, because $P(\alpha,\beta|\eta)$ is nonlinear in η (see figure 2a).

As seen in figure 2b the induced distribution of $P(\alpha,\beta)$ can be heavily skewed, making the expectation a possibly misleading choice. The median of the distribution is a more robust alternative. Of further interest are the minimum and maximum which can be seen as extreme quantiles. The extrema of the distribution limit the range of the low income proportion and thus give bounds on the best and worst possible amount of relative poverty given a range of equivalence weights. Let $Q_P(\gamma)$ be the γth quantile of the distribution of $P(\alpha,\beta)$, with $Q_P(0)$ being the minimum and $Q_P(1)$ being the maximum. It is defined as in equation (2) with $F_{Y^*}(z)$ replaced by $F_P(z)$, the cdf of the induced distribution of $P(\alpha,\beta)$. $F_P(z)$ can be written as

$$F_P(z) = \int_{\varepsilon_l}^{\varepsilon_u} \mathbb{1}(P(\alpha,\beta|u) \leq z) f_\eta(u) du, \tag{11}$$

where ε_l and ε_u are the endpoints of the support of f_η. Practical calculations can proceed in a simple fashion: $P(\alpha,\beta|\eta)$ is calculated on the support of f_η, the subset of results for which $P(\alpha,\beta|\eta) \leq z$ is selected, and the corresponding values of f_η are used to calculate $F_P(z)$.

In case of the results shown in figure 2b the mean equals 0.176. The minimum of the results in figure 2b is 0.154, the maximum is 0.229, and the median is about 0.165. Using the modified OECD scale yields a low income proportion of 0.162. This shows that reporting only a single point estimate based on a certain equivalence scale may be misleading, if one is not able to assume a specific equivalence scale as fixed. However, sensitivity analysis would usually stop at this point. As in most analyses sample data will be used, this ignores variability due to sampling. That is, the numbers just quoted may give an indication of the variability of $P(\alpha,\beta)$ with respect to equivalence scales, but only conditional on the data.

5.2 Variance estimation

Following Deville, 1999, the variance of E(P) can be calculated via its influence function.

Proposition 1. *Given an arbitrary univariate density f_η, the influence function of* E(P) *is given by*

$$I_{\mathrm{E}(P)}[M;k] = \int I_{P(\alpha,\beta|u)}[M; y_k/h_k^u] f_\eta(u) du.$$

This follows directly from rule 4 of Deville, 1999. The pseudo-variable z_k to be calculated for each observation k is simply given by the expectation of the pseudo-variables calculated via equation (8) for all values of η.

The influence function of $F_P(z)$ as defined in equation (11), which is required for the influence function and variance estimate of $Q_P(\gamma)$, can not be easily defined, because adding an observation may change the values of the indicator function in equation (11). To arrive at a simple solution, the following assumption is invoked: The low income proportion is a strictly decreasing function of η on the interval $[\varepsilon_l, \varepsilon_u]$, i.e. $\delta P(\alpha,\beta,\eta)/\delta\eta < 0$. Given this assumption it follows that

$$F_P(z) = \int_{\tau(z)}^{\varepsilon_u} f_\eta(u) du = 1 - F_\eta(\tau(z)), \tag{12}$$

where $\tau(z)$ is the smallest value in the interval $[\varepsilon_l, \varepsilon_u]$ for which $P(\alpha,\beta|\tau(z)) \leq z$ holds. It is a function of the data and does not necessarily equal ε_l. Using this simplified version of $F_P(z)$ it is possible to express $\tau(z)$ as the inverse of the low income proportion with α and β fixed, as each value of the

[4] Note that given a specific choice of η, i.e. a non-random scalar, the low income proportion for the population, $P(\alpha,\beta)$, is still a non-random scalar, too. The distribution of $P(\alpha,\beta)$ is solely due to the distribution of η, f_η.

[5] In the example shown in figure 2a, the correlation between η and $P(\alpha,\beta|\eta)$ is about -0.93.

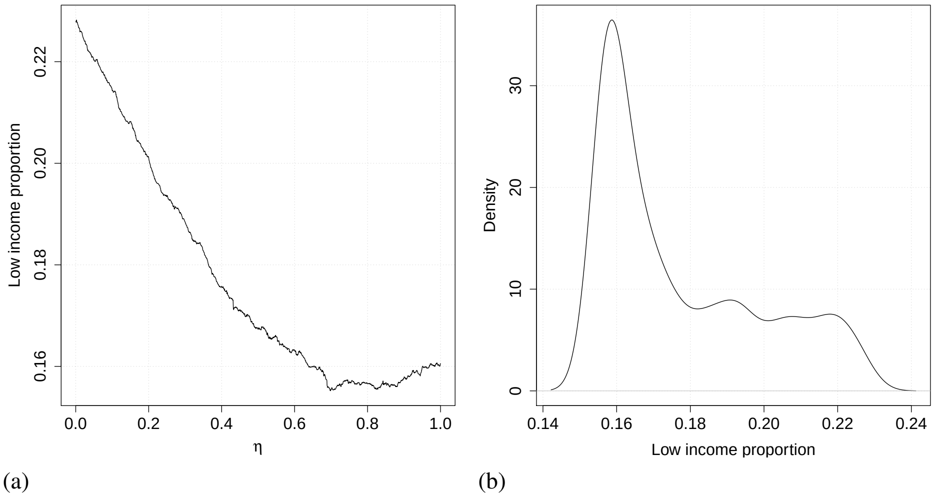


Figure 2. (a) Low income proportion as a function of η and (b) $f_{\hat{P}}$, both for the 2012 CNEF-file of the German Socio-Economic Panel using $\eta \sim \text{Unif}(0, 1)$.

low income proportion corresponds to exactly one value of η. This allows to calculate the influence function of $\tau(z)$ which otherwise would be hard to obtain. Given the simplified version of $F_P(z)$ it can be shown that the following proposition holds.

Proposition 2. *The influence function of $Q_P(\gamma)$ can be calculated as $I_{Q_P(\gamma)}[M;k] = I_{P(\alpha,\beta|\eta(\gamma))}[M; y_k/h_k^{\eta(\gamma)}]$ where $\eta(\gamma)$ is the value of η for which $Q_P(\gamma)$ results.*

For a proof see the appendix. From proposition 2 a simple two-step procedure follows. First, given f_η, calculate $F_P(z)$ and find $Q_P(\gamma)$ and the corresponding $\eta(\gamma)$. Second, use $\eta(\gamma)$ to calculate equivalent income and apply the standard formula (8) for variance estimation of the low income proportion.

The assumption invoked above may only hold approximately and only on a subset of the interval $[0, 1]$. For example, in case of the results shown in figure 2a, it is only satisfied on the interval $[0, 0.62]$. Given an income distribution it is hard to say whether the assumption will be satisfied, as the impact of η on the low income proportion via equations (6) and (5) is not easy to predict. Nevertheless, empirical evidence suggests that the relation between η and the low income proportion generally follows the pattern shown in 2a (Banks & Johnson, 1994; Burkhauser et al., 1996; Coulter et al., 1992). If the support of f_η is chosen correspondingly, this will not pose a problem. Furthermore, simulation results in the next section and in the supplementary materials show that variance estimates are only slightly biased if $[0, 1]$ is chosen as the support of f_η. While one should be cautious about concluding that violation of the assumption is unproblematic, it at least seems to be the case for the data used in this paper.

Note that proposition 2 holds for any measure of poverty and inequality for which the assumption of strict monotonicity is reasonable, i.e. indices need either to be strictly increasing or strictly decreasing in η. Further note that the distribution f_η is only needed at the first step of the procedure outlined above. As long as one is able to calculate the induced distribution of $P(\alpha, \beta)$ any distributional assumptions for η are possible. If only $Q_P(0)$ and $Q_P(1)$ have to be estimated then by the monotonicity assumption it suffices to specify the endpoints of the support of f_η. If the assumption is not satisfied the resulting values of $Q_P(0)$ and $Q_P(1)$ may be misleading, though.

As indicated above, $Q_P(0.5)$, $Q_P(0)$ and $Q_P(1)$ are proposed as quantiles of special interest. Simultaneous confidence intervals for $Q_{\hat{P}}(0)$ and $Q_{\hat{P}}(1)$ can be constructed using the Bonferroni correction. Let q_0^l be the resulting lower bound for $Q_{\hat{P}}(0)$ and let q_1^u be the resulting upper bound for $Q_{\hat{P}}(1)$ for the desired overall confidence level. q_0^l and q_1^u together yield an interval which captures both sampling variance and variability of equivalence scales. As such, it is proposed as an indication of overall uncertainty. The more willing one is to make strong assumptions about equivalence scales, the smaller this interval will become. For example, if we assume η as used in equation (6) to follow a uniform distribution $\text{Unif}(0.32, 0.72)$ will give a larger difference $q_1^u - q_0^l$ than assuming $\text{Unif}(0.50, 0.58)$, because equivalence scales "close" to each other usually lead to quite similar results. Assuming a single value for η can be seen as an extreme case, which only allows for sampling variance.

6 Monte carlo simulations

6.1 Simulation setup

To assess finite sample performance of the variance estimators proposed in the preceding section Monte Carlo simulations were run using the 2012 cross-national equivalent file (CNEF) of the German Socio-Economic Panel (SOEP), 2012.[6] The SOEP is a panel survey conducted annually since 1984 and covers a broad range of topics. CNEF data include a subset of variables from the SOEP, which are directly comparable to the CNEF versions of other surveys, easing replications. For a general description of the SOEP see Wagner, Frick, and Schupp, 2007 and for the CNEF see Frick, Jenkins, Lillard, Lipps, and Wooden, 2007.

The CNEF data include information on annual household income and the number of household members for a total of 12352 households.[7] Using this data as population frame, 50000 simple random samples without replacement were drawn for sampling fractions of 2.5%, 5%, and 10%, corresponding to sample sizes of 309, 618, and 1235, respectively. Additional results of simulations based on 50000 stratified random samples without replacement are described in appendix B in the supplementary materials and differ only slightly from the results presented here. Furthermore, simulation results for the Gini coefficient and the quintile share ratio can be found in appendix C, also in the supplementary materials. All programs are available from the author upon request.

For each of the 50000 samples, the standard errors of the following quantities were calculated: the low income proportion using the modified OECD scale and the mean, median, minimum and maximum of the induced distribution of the low income proportion. α and β were set to 0.6 and 0.5, thus following practice of Eurostat. Three variants were specified for f_η: one using Unif$(0.34, 0.51)$, one using Unif$(0.32, 0.72)$, and a variant using Unif$(0, 1)$, where the first variant uses only estimates taken from the survey of Dudel et al., 2013 which are based on expenditure data and demand systems. Using Unif$(0, 1)$ is rather extreme, as values close to 0 and 1 are not very sensible a priori and are not supported by the literature on equivalence scales. Additional simulations using a truncated normal distribution are presented in the supplementary materials. The induced distribution of the low income proportion was estimated by calculating $P(0.6, 0.5|\eta)$ for each percentile of f_η. The resulting distribution of values of $P(0.6, 0.5|\eta)$ was then used to estimate the mean and the quantiles of the induced distribution. Given a value of η $P(0.6, 0.5|\eta)$ was estimated by first generating equivalent weights for each household type according to equation (6) which were then plugged into equation (5).

Each of the estimates listed above is compared to its true value in terms of relative bias. Relative bias is calculated as $(\mathrm{E}(X_{\mathrm{sim}}) - X_{\mathrm{true}})/X_{\mathrm{true}}$, where $\mathrm{E}(X_{\mathrm{sim}})$ is the mean of the quantity of interest over all 50000 Monte Carlo repetitions and X_{true} is the true value. The true variance is calculated using the Monte Carlo variance, i.e. the variance of the estimators based on 50000 replications.

Density estimation as needed to calculate the influence function of the low income proportion as given by equation (8) was carried out using Gaussian kernels. Bandwidth h was calculated as $0.79(\hat{Q}_{Y^*}(0.75) - \hat{Q}_{Y^*}(0.25))n^{-0.2}$ as suggested by Silverman, 1986 for distributions with positive skew (see also Verma & Betti, 2005). $\hat{Q}_{Y^*}(\cdot)$ denotes the estimated quantile function of equivalent income as defined in section 2.1.

6.2 Results

Results on relative bias are shown in table 1. It is divided in four parts, each part corresponding to one of the different assumptions on the distribution of η. The first part includes results for the low income proportion calculated by approximating the modified OECD scale with $\eta = 0.54$. The other parts cover results for the expectation of the induced distribution, its median, minimum, and maximum, respectively. Tables with more detailed results can be found in appendix B in the supplementary materials.

As can be seen from table 1, estimates of standard errors based on the induced distribution generally can exhibit both upward and downward bias. Nevertheless, in most cases bias is negligible.

More specifically, bias is small in case of Unif$(0.34, 0.51)$, which is the only variant for which the assumption of monotonicity is plausible. Bias is even smaller for some cases if Unif$(0.32, 0.72)$ is used. An exception is the median, for which bias is high compared to minimum and maximum. It is still small in absolute terms, especially for larger sample sizes. Results for the third variant using $\eta \sim$ Unif$(0, 1)$ are somewhat more mixed. Bias is small for the expectation and the maximum, but relatively large for the median and the minimum. It decreases with sample size, though, and 95% confidence intervals (not shown) still are not too far off with coverage probabilities between 93% and 97%. In summary, variance estimation as proposed in section 5 works quite well, even if the monotonicity assumption is violated.

7 Empirical application

7.1 Data

In this section empirical results for Germany will be presented which make use of the framework outlined in section 5. CNEFs from the SOEP covering the years 2000 up to 2012 will be used. For each year the same quantities are calculated: the low income proportion based on the modified OECD scale and the median of the induced distribution of the low income proportion using each Unif$(0.34, 0.51)$

[6]The data can be obtained from the German Institute for Economic Research, Berlin. See http://www.diw.de/en/soep. The DOI of the data set is: 10.5684/soep.v29

[7]Note that unit and item nonresponse and other reasons of attrition are not considered. Because the CNEF already includes imputed values all 12352 households can be used for simulations.

Table 1
Relative bias, simple random sampling

Sampling fraction	2.5% (n = 309)	5% (n = 618)	10% (n = 1235)
		$\eta = 0.54$	
SE(P)	−0.003	−0.006	−0.009
		$\eta \sim$ Unif(0.34, 0.51)	
SE(E(P))	−0.011	−0.015	−0.001
SE(Q_P(0.5))	0.009	0.003	−0.003
SE(Q_P(0))	0.014	0.013	0.013
SE(Q_P(1))	−0.004	−0.003	−0.003
		$\eta \sim$ Unif(0.32, 0.72)	
SE(E(P))	−0.012	−0.015	−0.001
SE(Q_P(0.5))	0.049	0.028	0.018
SE(Q_P(0))	0.002	0.001	0.005
SE(Q_P(1))	0.020	0.007	0.001
		$\eta \sim$ Unif(0, 1)	
SE(E(P))	−0.016	−0.020	−0.006
SE(Q_P(0.5))	0.082	0.066	0.061
SE(Q_P(0))	−0.087	−0.061	−0.037
SE(Q_P(1))	−0.009	−0.015	−0.018

and Unif(0.32, 0.72). Additionally, confidence intervals are calculated as well as the simultaneous confidences intervals of the minimum and maximum of the induced distributions. Moreover, in addition to the modified OECD scale two alternative equivalence scales were used. One is the so called square-root method proposed by the OECD, which calculates equivalence scales as $A_h = \sqrt{h}$. The other equivalence scale is an approximation to the scale published by Koulovatianos, Schröder, and Schmidt, 2005 and uses $\eta = 0.72$.

The sampling design of the SOEP is rather intricate and the data include design weights and cross-sectional raking weights, both of which will be used for analysis. Note that the data also include weights to account for panel attrition. These were not used in the calculation of the results presented below, though. An extended discussion of weighting such a complex sample as the SOEP and additional results of calculations accounting for panel attrition can be found in appendix D in the supplementary materials.

For cross-sectional weights it has to be taken into account that these depend on the sample. Deville, 1999 derived a procedure for this case that runs as follows (see also Berger & Skinner, 2003; Graf & Tillé, 2014). Let $\mathbf{X}$ denote the variables which are used for weighting and $\mathbf{x}_k$ the observed values of these variables for observation k. w_k is the cross-sectional weight which results for observation k. For variance estimation the linearized variable

$$\tilde{z}_k = \hat{z}_k - \mathbf{x}_k'\hat{\boldsymbol{\beta}} \qquad (13)$$

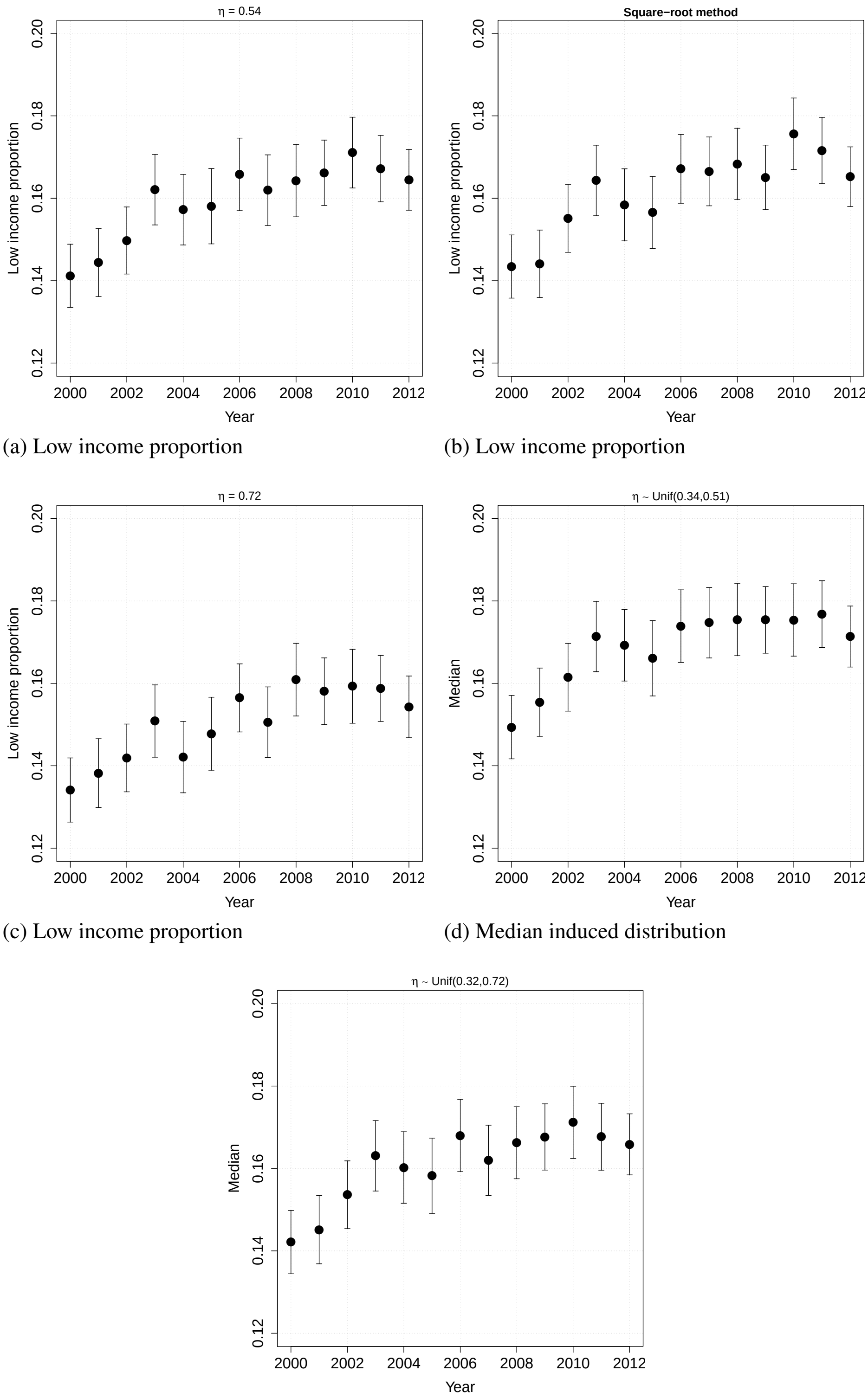
is used. $\hat{\boldsymbol{\beta}}$ is the estimated vector of coefficients of the weighted regression of $\mathbf{x}_k$ on $\hat{z}_k$, weighted by w_k. Thus, $\tilde{z}_k$ is defined as the residual of k in this regression. The following variables were used: state; household size; home ownership status; size of community. Variance estimation is based on the variance of $\tilde{t}_{\tilde{z}} = \sum_{k \in S} d_k \tilde{z}_k$, where d_k is the design weight. The variance of $\tilde{t}_{\tilde{z}}$ is estimated using the Hájek, 1964 approximation of the Horvitz-Thompson estimator.

7.2 Results

Results can be found in figures 3a to 4b. Figure 3a shows point estimates and 95% confidence intervals of the low income proportion using $\eta = 0.54$, i.e. the modified OECD scale. Figures 3b and 3c show results obtained by setting $\eta = 0.5$ (square root method) and $\eta = 0.72$ (Koulovatianos et al., 2005), respectively. Figures 3d and 3e show point estimates and confidence intervals of the median of the induced distributions using $\eta \sim$ Unif(0.34, 0.51) and $\eta \sim$ Unif(0.32, 0.72), respectively.

Not surprisingly, differences between results of the modified OECD scale (figure 3a) and of the square-root method (figure 3b) are negligible with respect to both the point estimates and the confidence intervals. The scale of Koulovatianos et al., 2005 (figure 3c) results in the low income proportion being approximately one percentage point lower than in case of the modified OECD scale, pointing again to the sensitivity of the low income proportion to the choice of η. The width of confidence intervals and the general trend of the low income proportion over time are similar, though.

Results of the modified OECD scale (figure 3a) and the median of the distribution induced by assuming $\eta \sim$



(a) Low income proportion

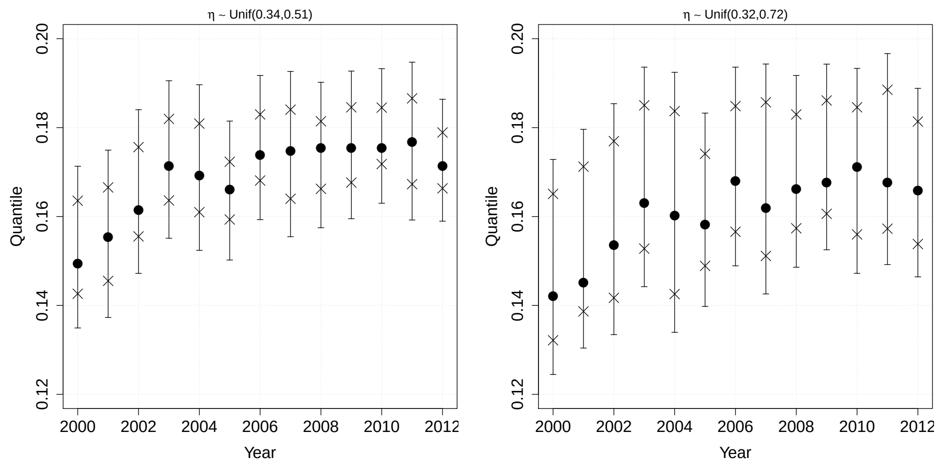
(b) Low income proportion

(c) Low income proportion

(d) Median induced distribution

(e) Median induced distribution

Figure 3. Point estimates and confidence intervals using SOEP CNEF data.



(a) (b)

Figure 4. Point estimate of the median and simultaneous confidence interval of minimum and maximum ($\eta \sim \text{Unif}(0.34, 0.51)$ and $\eta \sim \text{Unif}(0.32, 0.72)$) using SOEP CNEF data. Point estimates of minimum and maximum are shown as crosses.

Unif(0.34, 0.51) (figure 3d) exhibit some differences. For example, the point estimate of the median of the induced distribution amounts to 0.175 for 2010 and 0.177 for 2011, whereas the use of the modified OECD scale results in values of 0.171 and 0.167, respectively. Moreover, trends in point estimates differ. Using the modified OECD scale the low income proportion increases between 2007 and 2010 by 0.009, while the difference between these years is less than 0.001 in case of the median of the induced distribution. Generally, the mean absolute difference of point estimates is about 0.009. Upper and lower limits of confidence intervals are shifted by a similar amount, but the width of the confidence intervals, i.e. the difference between upper and lower limit of the confidence intervals, is quite similar in both cases and amounts to 0.017.

If the support of η is extended to the interval $[0.32, 0.72]$ differences as compared to the modified OECD scale are less pronounced (figure 3e). For example, using the modified OECD scale the low income proportion amounts to 0.171 for the year 2010 and the median of the induced distribution also amounts to 0.171. Furthermore, the width of the confidence intervals is quite similar and differences are small.

Figures 4a and 4b show the median of the induced distributions, but the confidence interval of the median has been replaced with the lower and upper bound of the simultaneous 95% confidence interval of the minimum and the maximum, calculated using the Bonferroni correction as proposed in section 5.[8]. As can be seen in both figures this interval is not necessarily symmetric. The width of the confidence intervals is now much larger. For instance, for 2010 the width amounts to 0.030 (figure 4a) and 0.046 (figure 4b), respectively, whereas it amounts to 0.018 in the other cases (figures 3d and 3e). Generally, the mean ratio of widths as compared to the confidence intervals for the low income proportion using the modified OECD scale is 2.05 and 2.86, respectively. This also holds if one uses the square root method or the scale of Koulovatianos et al., 2005 instead of the modified OECD scale.

The total width of the confidence intervals in figures 4a and 4b can be decomposed into two parts: The contribution of equivalence scale uncertainty and the variability due to sampling, where the former is given by the difference of the maximum and the minimum of the induced distribution, while the latter is calculated as the sum of the difference between the upper endpoint of the confidence interval and the maximum and the difference between the minimum and the lower endpoint. For example, the total width of 0.030 for 2010 seen in figure 4a can be decomposed into 0.013 (equivalence scale uncertainty) and 0.018 (sampling variance). Overall, between 42% and 57% of the interval widths shown in figure 4a are due to equivalence scale uncertainty and 58% to 70% in figure 4b.

These results have several implications. First, and most importantly, they show that much variability is ignored by using only a single equivalence scale, even compared to the relatively restrictive case of using Unif(0.34, 0.51). Estimates derived by using a fixed equivalence scale are still valid conditional on the specific scale, though. Second, the range of intervals heavily depends on the distribution assumed for η. This shows that assumptions of sensitivity analysis should be well-founded and not be chosen arbitrarily. Third, choosing a relatively small range of possible values for η may lead to small intervals, but may not be consistent with the

[8]Calculation proceeds by assuming normality and estimating 97.5% confidence intervals for maximum and minimum to achieve a joint confidence level of at least 95%.

equivalence scales used for main analysis, as is suggested by the comparison of the results of the modified OECD scale and equivalence scales based on demand systems. Using the framework presented in this paper and reporting the median or mean of the induced distribution as main estimates could avoid such inconsistencies. Either way, this again highlights that both the choice of equivalence scale for main analysis and the assumptions for sensitivity analysis should not be taken lightly.

8 Conclusion

In this paper a method for acknowledging variability of equivalence scales in the estimation of indices of poverty and inequality was proposed, which can be uses as a formal framework for sensitivity analysis. Starting from a simple parametrization of equivalence scales the method is based on subjective assumptions about the distribution of equivalence scales made by the researcher. These assumptions lead not to a single estimate, but to an induced distribution of a measure of poverty or inequality. It was shown that variance estimates for parameters of the induced distribution can be calculated using standard approaches. The use of the simultaneous confidence interval of the extrema of the induced distribution was proposed as an useful addition to standard confidence intervals.

Results of simulations show that the approach leads to reliable variance estimates. As an example the approach was applied to data from the German Socio-Economic Panel, demonstrating that the procedure leads to a much broader range of results than the standard approach. Using only a certain equivalence scale, e.g. the modified OECD scale, leads to results which underestimate uncertainty and which rest on strong assumptions. While results conditional on a fixed equivalence scale are still valid, conducting sensitivity analysis and estimating the variance of the results of sensitivity analysis as proposed in this paper helps to avoid these issues.

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Appendix
Proof of proposition 2

The influence function of I_{Q_P} can be derived in the following fashion. Because $F_P(Q_P(\gamma)) = \gamma$ is constant, the influence function of F_P equals zero. Using rule 7 of Deville, 1999, which is similar to taking the total differential, this can be expanded to

$$I_{F_P}[M;k] = I_{F_P}[M;k|Q_P(\gamma)] + f_P(Q_P(\gamma))I_{Q_P}(M;k) = 0, \quad (14)$$

where $I_{F_P}[M;k|Q_P(\gamma)]$ is the influence function of F_P with $Q_P(\gamma)$ fixed at the current value. Rearranging gives

$$I_{Q_P}[M;k] = -\frac{1}{f_P(Q(\gamma))} I_{F_P}[M;k|Q(\gamma)]. \quad (15)$$

The influence function I_{F_P} is given by

$$I_{F_P}[M;k|Q_P(\gamma)] = -f_\eta(\tau(Q_P(\gamma)))I_{\tau(Q_P(\gamma))}[M;k], \quad (16)$$

which requires the influence function $I_{\tau(Q_P(\gamma))}[M;k]$. Given the assumption introduced in section 5.2 $\tau(z)$ can be defined in terms of the inverse of $P(\alpha,\beta,\eta)$ with respect to η and α and β fixed. Let $\tau(z) = P^{-1}(z)$ be this inverse function which gives for any value of the low income proportion p the corresponding value of η such that if $P(\eta) = p$ then $P^{-1}(p) = \eta$. Given α and β $P(P^{-1}(Q_P(\gamma))) = p$ is constant from which

$$I_{\tau(Q_P(\gamma))}[M;k] = -\frac{1}{\delta P(\eta)/\delta\eta|\eta(\gamma)} I_P[M;k;\eta(\gamma)] \quad (17)$$

follows, where $\eta(\gamma)$ is the result of $P^{-1}(Q_P(\gamma))$, i.e. the value of η which leads to the γth quantile of the distribution of $P(\alpha,\beta)$. $f_P(Q_P(\gamma))$ in equation (15) can be written as

$$f_P(Q_P(\gamma)) = f_\eta(\eta(\gamma))\frac{1}{\delta P(\eta)/\delta\eta|\eta(\gamma)}. \quad (18)$$

Plugging equations (16), (17), and (18) back in equation (15) gives the desired result. Note that this proof does not depend on the definition of the low income proportion which can be replaced with any other measure of interest. The proof for the case of a measure strictly increasing in η follows in the same way.