# MATHEMATICAL MODEL OF THE SKATEBOARD

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In the present paper we construct the mathematical model, describing the motion of the rider on a skateboard. The motion of the model is described in the absence of rider control. The equations of motion of the model are obtained in the form of Gibbs-Appell equations, and their stability analysis is fulfilled. The influence of different parameters of the model on its dynamics is investigated.

**KEY WORDS:** skateboard, nonholonomic system, Gibbs-Appell equations, stability of motion.

**INTRODUCTION:** At the present time skateboarding – the art of riding on the skateboard – is one of the most popular recreational sports. Nevertheless, serious researches concerning dynamics and stability of the skateboard motion are almost absent. In the late 70ies - early 80ies of the last century Mont Hubbard (Hubbard, 1979 and 1980) proposed several mathematical models describing the motion of the rider on the skateboard. To derive the equations of motion of the models he used the principal theorems of dynamics. In this work we present a further development of the models offered by Hubbard, using the equations of motion in Gibbs-Appell form. Besides the investigations by Hubbard it is also necessary to mention the paper of Ispolov & Smolnikov (1996) and the recent paper of Wisse & Schwab (2005), devoted to study of various mathematical models of a skateboard. However the model proposed by Ispolov & Smolnikov (1996), is two-dimensional while in the papers of Hubbard (1979 and 1980) a more realistic three-dimensional model is studied. As to the paper of Wisse & Schwab (2005), it contains only the brief review of the main results obtained in Hubbard (1979).

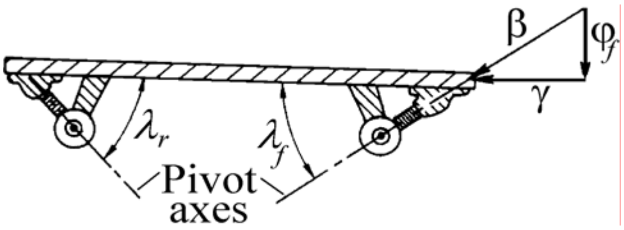


Fig.1.

The usual skateboard consists of a board, two trucks, and four wheels (Fig. 1). The modern boards are usually from 78 to 83 cm long, 17 to 21 cm wide and 1 to 2 cm thick. Beside the dimensions, the boards differ in bending stiffness, depending on the particular application desired. If the basic rider purpose is the opportunity to execute difficult tricks, it is better to use a more flexible board. More rigid board should be used at high speed riding. The most essential elements of a skateboard are the trucks, connecting the axles to the board. Angular motion of both the front and rear axles is constrained to be about their respective nonhorizontal pivot axes, thus causing a steering angle of the wheels whenever the axles are not parallel to the plane of the board. The vehicle is steered by using this kinematic relationship between steering angles and tilt of the board. In addition, there is a torsional spring, which exerts a restoring torque between the wheelset and the board proportional to the tilt of the board with respect to the wheelset.

**METHOD AND RESULTS:** Let us assume that the rider, modelled as a rigid body, remains perpendicular with respect to the board. Therefore, when the board tilts through $\gamma$, the rider tilts through the same angle relative to vertical. Let us introduce an inertial coordinate system $OXYZ$ in the ground plane. Let $AB = a$ is a distance between centres of two axles of a skateboard. The position of a line $AB$ with respect to $OXYZ$ system is defined by $X$ and $Y$ coordinates of its centre and by the angle $\theta$ between this line and the $OX$ -axis (Fig. 2). The tilt of the board causes the rotation of front wheels clockwise through $\varphi_f$ and the rotation of rear wheels anticlockwise through $\varphi_r$ (Fig. 3). The wheels of the skateboard are assumed to roll without lateral sliding. This condition is modelled by constraints, which may be shown to be nonholonomic

$$-\dot{X}\sin\left(\theta-\varphi_f\right)+\dot{Y}\cos\left(\theta-\varphi_f\right)+\frac{a}{2}\dot{\theta}\cos\varphi_f=0,\qquad -\dot{X}\sin\left(\theta+\varphi_r\right)+\dot{Y}\cos\left(\theta+\varphi_r\right)-\frac{a}{2}\dot{\theta}\cos\varphi_r=0.$$

Under these conditions the velocities of points $A$ and $B$ will be directed horizontally and perpendicularly to the axles of wheels, and there is a point $C$ on the line $AB$ which has zero lateral velocity. Its forward velocity we denote by $u$. It may be shown, that

$$AC=\frac{a\sin\varphi_f\cos\varphi_r}{\sin\left(\varphi_f+\varphi_r\right)},\quad u=-\frac{a\dot{\theta}\cos\varphi_f\cos\varphi_r}{\sin\left(\varphi_f+\varphi_r\right)},\ \text{i.e. }\dot{\theta}=-\frac{u\sin\left(\varphi_f+\varphi_r\right)}{a\cos\varphi_f\cos\varphi_r}. \tag{1}$$

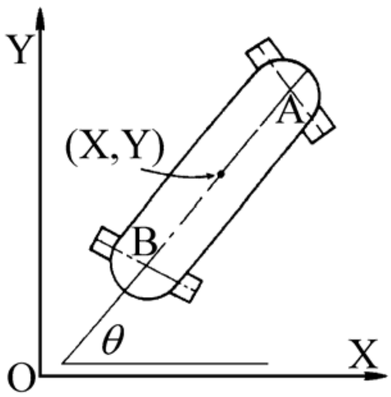


Fig. 2.

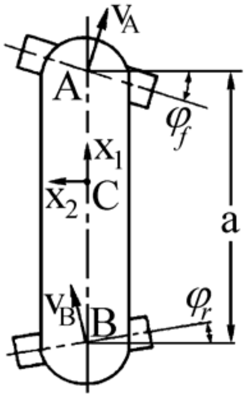


Fig. 3.

The equations of motion of the skateboard will be written using the $Cx_1x_2x_3$ coordinate system whose origin is at $C$. The $Cx_1$-axis is in the direction of motion; the $Cx_3$-axis is up and $Cx_2$ is to the left in the horizontal plane (Fig. 3). Let us denote the unit vectors of this coordinate system by $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$. We will assume, basing on the features of a skateboard design, that the angles $\theta$, $\varphi_f$ and $\varphi_r$ are small enough and

$$\cos\gamma\approx 1,\ \cos\varphi_f\approx 1,\ \cos\varphi_r\approx 1,\ \sin\gamma\approx\gamma,\ \sin\varphi_f\approx\varphi_f,\ \sin\varphi_r\approx\varphi_r.$$

Taking into account these assumptions, we can rewrite expressions (1) in the form:

$$AC=\frac{a\varphi_f}{\varphi_f+\varphi_r},\qquad u=-\frac{a\dot{\theta}}{\varphi_f+\varphi_r},\qquad \dot{\theta}=-\frac{u\left(\varphi_f+\varphi_r\right)}{a}. \tag{2}$$

Let us find now the relationship between the steering angles and tilt of the board. For this purpose we will use the theory of infinitesimal rotations (Synge & Griffith, 1959). Assume an infinitesimal rotation $\boldsymbol{\beta}$ of the forward axle about its pivot axle. Then the wheels rotate through $-\gamma$ about $Px_1$ and $-\varphi_f$ about $Px_3$ (Fig. 1). Taking into account that the resultant of two infinitesimal rotations about the same point is the vector sum of those rotations, we obtain $\boldsymbol{\beta}=-\gamma\mathbf{e}_1-\varphi_f\mathbf{e}_3$, and therefore

$$\varphi_f=\gamma\tan\lambda_f. \tag{3}$$

Similarly, for the rear truck

$$\varphi_r=\gamma\tan\lambda_r. \tag{4}$$

Using constraints (3)-(4) we can rewrite expressions (2) as follows:

$$AC=\frac{a\tan\lambda_f}{\tan\lambda_f+\tan\lambda_r},\quad \dot{\theta}=-\frac{u\gamma\left(\tan\lambda_f+\tan\lambda\right)}{a}.$$

Let us assume, that the board of the skateboard is located on the distance $h$ above the line $AB$. The length of the board is also equal to $a$. The board's center of mass $G$ is located in its center. Then the absolute velocity and absolute acceleration of the board's center of mass may be written as follows:

$$\mathbf{V}_G=u\mathbf{e}_1-\left[\frac{u\gamma}{2}\left(\tan\lambda_f-\tan\lambda_r\right)+h\dot{\gamma}\right]\mathbf{e}_2,$$

$$\mathbf{W}_G = \dot{u}\mathbf{e}_1 - \left[ \frac{u^2\gamma}{a}\left(\tan\lambda_f + \tan\lambda_r\right) + \frac{u\dot{\gamma} + \dot{u}\gamma}{2}\left(\tan\lambda_f - \tan\lambda_r\right) + h\ddot{\gamma} \right]\mathbf{e}_2.$$

We will find now the absolute velocity and absolute acceleration of the rider's center of mass. For more generality we suppose, that the rider's center of mass $R$ is not located above the board center of mass, but it is located over the longitudinal axis of the board on a distance $b$ from the front truck. Let $\ell$ be the height of the rider's center of mass above point $C$. Then the absolute velocity and absolute acceleration of the rider's center of mass may be written as follows:

$$\mathbf{V}_R = u\mathbf{e}_1 - \left[ {}^{u\gamma}\!/_{a}\left((a-b)\tan\lambda_f - b\tan\lambda_r\right) + \ell\dot{\gamma} \right]\mathbf{e}_2,$$

$$\mathbf{W}_R = \dot{u}\mathbf{e}_1 - \left[ \frac{u^2\gamma}{a}\left(\tan\lambda_f + \tan\lambda_r\right) + \frac{u\dot{\gamma} + \dot{u}\gamma}{a}\left((a-b)\tan\lambda_f - b\tan\lambda_r\right) + l\ddot{\gamma} \right]\mathbf{e}_2.$$

The absolute angular velocity and angular acceleration of the board and the rider can be represented as:

$$\mathbf{\Omega} = \dot{\gamma}\mathbf{e}_1 + \dot{\theta}\mathbf{e}_3 = \dot{\gamma}\mathbf{e}_1 - \frac{u\gamma}{a}\left(\tan\lambda_f + \tan\lambda_r\right)\mathbf{e}_3, \quad \dot{\mathbf{\Omega}} = \ddot{\gamma}\mathbf{e}_1 - \frac{\dot{u}\gamma + u\dot{\gamma}}{a}\left(\tan\lambda_f + \tan\lambda_r\right)\mathbf{e}_3.$$

Now we can obtain the equations of motion of the given model of a skateboard in the form of Gibbs-Appell equations (Lewis, 1996). We can choose the variables $u$ and $\dot{\gamma}$ as a pseudovelocities for this problem. As a first step, we should find the Appell function (energy of acceleration). Denote the mass of the board by $m_b$, the mass of the rider by $m_r$. Let $I_b$ be the moment of inertia of the board with respect to the axis, passing through its center of mass, parallel to the $Px_1$-axis. And let $I_r$ be the moment of inertia of the rider with respect to the axis, passing through its center of mass. For the Appell function we have the following expression:

$$S = \frac{m_b + m_r}{2}\dot{u}^2 + \frac{I_b + I_r + m_b h^2 + m_r\ell^2}{2}\ddot{\gamma}^2 + \frac{u^2}{2}\left(m_b h + m_r\ell\right)\left(\tan\lambda_f + \tan\lambda_r\right) +$$
$$+\left[ \frac{m_b h}{2}\left(\tan\lambda_f - \tan\lambda_r\right) + \frac{m_r\ell}{a}\left((a-b)\tan\lambda_f - b\tan\lambda_r\right) \right]\left(\dot{u}\gamma + u\dot{\gamma}\right)\ddot{\gamma}.$$

The potential energy of the system consists of the potential due to gravity and the potential due to the torsional spring. Therefore,

$$V = \frac{k\gamma^2}{2} + \left(m_b h + m_r\ell\right)\mathrm{g}\cos\gamma \approx \left(m_b h + m_r\ell\right)\mathrm{g} + \left(k - \mathrm{g}\left(m_b h + m_r\ell\right)\right)\frac{\gamma^2}{2}.$$

The Gibbs-Appell equations, describing the dynamics of the given model of a skateboard, may be written as follows:

$$\partial S\!/\!\partial\ddot{u} = 0, \quad \partial S\!/\!\partial\ddot{\gamma} = -\partial V\!/\!\partial\gamma$$

or, in explicit form

$$\ddot{u} = 0, \tag{5}$$

$$\left(I_b + I_r + m_b h^2 + m_r\ell^2\right)\ddot{\gamma} + \left[ \frac{m_b h}{2}\left(\tan\lambda_f - \tan\lambda_r\right) + \frac{m_r\ell}{a}\left((a-b)\tan\lambda_f - b\tan\lambda_r\right) \right]u\dot{\gamma} +$$
$$+\left[ \frac{u^2}{2}\left(m_b h + m_r\ell\right)\left(\tan\lambda_f + \tan\lambda_r\right) + k - \left(m_b h + m_r\ell\right)\mathrm{g} \right]\gamma = 0. \tag{6}$$

Equation (5) implies, that point $C$ has a constant forward speed $u$. Hence $u$ will be treated as a parameter in the equation (6).

Now let us make the stability analysis of the system. For stable motion all the coefficients in the equation (6) must be positive. The first coefficient is always positive; hence, the conditions of stability of motion have a following form:

$$\left[\frac{m_b h}{2}\left(\tan\lambda_f - \tan\lambda_r\right) + \frac{m_r \ell}{a}\left((a-b)\tan\lambda_f - b\tan\lambda_r\right)\right]u > 0, \tag{7}$$

$$\frac{u^2}{2}\left(m_b h + m_r \ell\right)\left(\tan\lambda_f + \tan\lambda_r\right) + k - \left(m_b h + m_r \ell\right)\mathrm{g} > 0. \tag{8}$$

We can make now simple conclusions using equation (6) and conditions (7)-(8). Note, first of all, that the stability of motion depends on its direction. If one direction of motion is stable, the opposite direction is necessarily unstable. If the speed $u$ is zero, motion of $\gamma$ is oscillatory and bounded in the case, where the torsional spring constant is sufficient to overcome the destabilizing gravity torque.

For the symmetric case, when $b = a/2$ and $\lambda_f = \lambda_r = \lambda,$ the $\dot{\gamma}$ coefficient is zero. In this case, we can note, that the nonzero forward speed $u$ can stabilize $\gamma$ motion. Indeed, when $k < \left(m_b h + m_r \ell\right)\mathrm{g}$, there is a critical forward speed

$$u_* = \sqrt{\frac{a\left(\left(m_b h + m_r l\right)g - k\right)\cot\lambda}{2\left(m_b h + m_r l\right)}},$$

above which $\gamma$ motion will be stable. Thus, it is theoretically possible to have a vehicle, which is initially unstable at zero speed and becomes stable at higher speeds due to inertia effects.

Let us assume, that $\lambda_f = \lambda_r = \lambda,\ b \neq a/2$ and condition (8) is valid. Then, from condition (7) we obtain, that the stability of $\gamma$ motion depends on the location of the rider. If the rider stands closer to the front truck $(b < a/2)$, the motion will be stable and when the rider stands closer to the rear truck $(b > a/2)$, the motion will be unstable.

**CONCLUSION:** In this study we construct the simple mathematical model describing the motion of the rider on the skateboard. The equations of motion for this model have been derived in the form of Gibbs-Appell equations and stability criteria presented. The stability of this system depends on the sign and the value of velocity of forward motion.

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